MATH 211 FINAL EXAM

1. **(a)** Let $f(x)$ be given by

$$f(x) = \begin{cases} 7\,x-5 & x<3 \\ x^2+2\,x-1 & x\geq 3 \end{cases}.$$

Calculate

$$\lim_{x\to 3^-} f(x)$$

Solution:

Approach from negative side so

$$\lim_{x\to 3^-} f(x) = 7\cdot 3-5 = 16$$

(b) Calculate

$$\lim_{x\to 4} \frac{x-4}{\sqrt{x}-2}$$

Solution:

$$\lim_{x\to 4} \frac{x-4}{\sqrt{x}-2} = \lim_{x\to 4} \frac{\left(\sqrt{x}-2\right)\left(\sqrt{x}+2\right)}{\sqrt{x}-2} = \lim_{x\to 4} \left(\sqrt{x}+2\right) = \left(\sqrt{4}+2\right) = 4$$

(c) Calculate

$$\lim_{x\to\infty} \frac{13-11\,x^2}{3-5\,x+7\,x^2}$$

Solution:

$$\lim_{x\to\infty} \frac{13-11\,x^2}{3-5\,x+7\,x^2} = \lim_{x\to\infty} \frac{\frac{13}{x^2}-11}{\frac{3}{x^2}-\frac{5}{x}+7} = -11/7$$

(d) Calculate

$$\lim_{x\to 2^+} \ln(x-2)$$

Solution:

$$\lim_{x\to 2^+} \ln(x-2) = \lim_{x\to 0^+} \ln(x) = -\infty$$

2. Calculate the derivative of the following functions. Simplify only if appropriate.

(a) $f(x) = (1 - x + x^5)(1 - x^2 + x^7)$

Solution:

$f'(x) = (-1 + 5x^4)(1 - x^2 + x^7) + (1 - x + x^5)(-2x^2 + 7x^6)$

(b) $g(x) = \frac{x^3+3x}{x^2-4x+3}$

Solution:

$g'(x) = \frac{(3x^2+3)(x^2-4x+3)-(x^3+3x)(2x^2-4)}{(x^2-4x+3)^2}$

(c) $h(x) = \sqrt{x^4 + 3x + 1}$

Solution:

$h'(x) = \frac{1}{2\sqrt{x^4+3x+1}} \cdot (4x^3 + 3)$

(d) $i(x) = 2\ln(x^3) - 3\ln(e^x) + \frac{1}{e^{2x}}$

Solution:

$i(x) = 6\ln(x) - 3x + e^{-2x}$ so $i'(x) = \frac{6}{x} + e^{-2x} \cdot (-2)$

3. Calculate the following integrals.

(a) $\int \left(x^2 - \sqrt{x} - \frac{1}{\sqrt{x}}\right) dx$

Solution:

$\int \left(x^2 - \sqrt{x} - \frac{1}{\sqrt{x}}\right) dx =$

$\int (x^2 - x^{1/2} - x^{-1/2})\, dx = \frac{x^3}{3} - \frac{2}{3}x^{3/2} - 2x^{1/2} + C = \frac{x^3}{3} - \frac{2}{3}x\sqrt{x} - 2\sqrt{x} + C =$

(b) $\int_1^4 \left(x - \sqrt{x}\right) dx$

Solution:

$\int_1^4 \left(x - \sqrt{x}\right) dx = \left[\frac{x^2}{2} - \frac{2}{3}x\sqrt{x}\right]_1^4 = \left(8 - \frac{16}{3}\right) - \left(\frac{1}{2} - \frac{2}{3}\right) = \frac{8}{3} + \frac{1}{6} = \frac{17}{6}$

(c) $\int_0^3 6x^2(x^3 + 2)^4\, dx$

Solution:

Let $u = x^3 + 2,\ du = 3x^2\, dx$

$\int_0^3 6x^2(x^3 + 2)^4\, dx = 2\int_2^{29} u^4\, du = \frac{2}{5}[u^5]_2^{29} = \frac{2}{5}\left(29^5 - 2^5\right) = 20\,511\,117/5$

(d) $\int\left(x^e - e^x - \frac{1}{x}\right) dx$

Solution:

$\int\left(x^e - e^x - \frac{1}{x}\right) dx = \frac{x^{e+1}}{e+1} - e^x - \ln(x) + C$

4. **(a)** Is the function

$$f(x) = \begin{cases} 3\,x - 1 & x < 2 \\ 5 & x = 2 \\ 8 - 2\,x & x > 2 \end{cases}$$

continuous? To get credit you must supply your reasoning.

Solution:

Since the left-sided limit at 2 is 5 and the right-sided at 2 limit is 4, there is no limit and hence the function is not continuous.

(b) What is the domain of the function $g(x) = \frac{3\,x-15}{2\,x^2-50}$?

Solution:

There is trouble if $2\,x^2 - 50 = 0$ so the domain must exclude $x = 5$ and $x = -5$.

(c) Find a continuous function $h(x)$ that is defined for all $x \geq 0$ and satisfies $h(x) = g(x)$ when $x \geq 0$.

Solution:

When $x \neq 5$ the following calculation is legitimate

$\frac{3\,x-15}{2\,x^2-50} = \frac{3}{2}\,\frac{(x-5)}{(x-5)\,(x+5)} = \frac{3}{2}\,\frac{1}{x+5}$ so choose $h(x) = \frac{3}{2}\,\frac{1}{x+5}$, which is continuous when $x \geq 0$.

5. Let $f(x) = \frac{x}{(1-x)^2} - 2$ so that $f'(x) = \frac{x+1}{(1-x)^3}$ and $f''(x) = \frac{2\,(x+2)}{(1-x)^4}$.

(a) Determine if there are any vertical and/or horizontal asymptotes. If so, write their equation(s).

Solution:

Vertical $x = 1$, horizontal $y = 2$.

(b) Find and classify all relative extrema.

Solution:

The only critical value is when $f'(x) = 0$ so $x = -1$. Since $f''(-1) > 0$, $x = -1$ is a relative minimum.

(c) Determine where $f(x)$ is concave up and whether there are any inflection points.

Solution:

Potential inflection only when $f''(x) = 0$ so $x = -2$. Concave up means $f''(x) > 0$ and this only happens when $x > -2$. It follows that $x = 2$ corresponds to an inflection.

(d) Determine all y-intercepts and x-intercepts.

Solution:

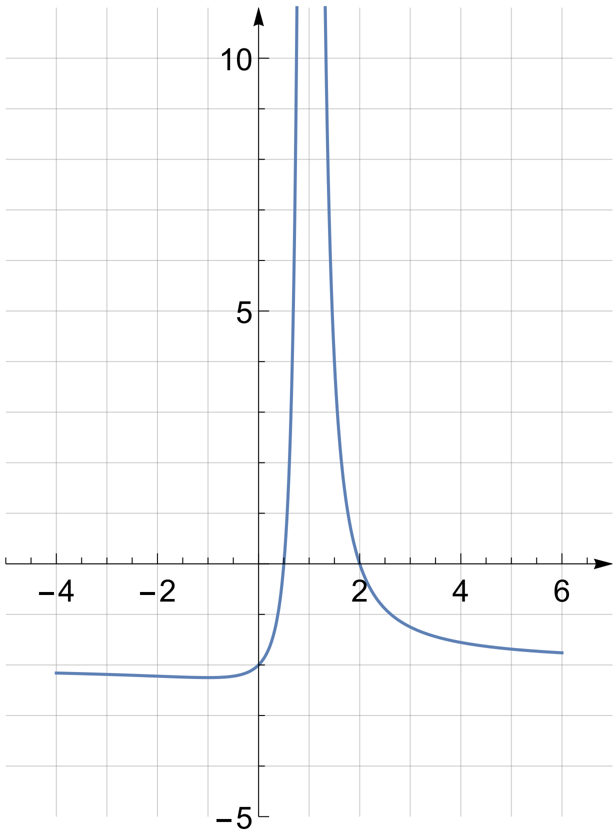
The y-intercept is $(0, -2)$.

The x-intercepts have $f(x) = 0$ so $\frac{x}{(1-x)^2} = 2$ or $x = 2 - 4x + 2x^2$ and $2x^2 - 5x + 2 = 0$.

Now $2x^2 - 5x + 2 = (x-2)(2x-1)$ so $x = 2$ and $x = \frac{1}{2}$.

(e) Graph $f(x)$ and make sure it satisfies all the properties discovered in **(a)-(d)**.

Solution:

5. 

6. **(a)** Determine the absolute minimum value of

$$f(x) = \frac{1}{4}x^4 - \frac{3}{2}x^2 - 2x + 3.$$

Hint: The derivative may be written $f'(x) = (x+1)^2(x-2)$.

Solution:

The derivative is zero only when $x = -1$ and $x = 2$.

Now $f'(x) > 0$ when $x > 2$, and $f'(x) < 0$ when $x < 2$.

The minimum is when $x = 2$.

(b) Determine the absolute maximum of $g(x) = 2\,x^3 - x^2 - 4\,x$ when x is restricted by $0 \leq x \leq 2$.

Solution:

The derivative is $g'(x) = 6\,x^2 - 2\,x - 4 = 2\,(3\,x^2 - x - 2) = 3\,(x-1)\,(3\,x+2)$ so only $x = 1$ is of interest as $x = -\frac{2}{3}$ does not satisfy the restrictions.

Now compare with the endpoints $g(0) = 0,\ g(1) = -3,\ g(2) = 4$.

The maximum is at $x = 2$.

7. An archaeological survey has produced a well-preserved specimen that has been sent to a laboratory for further testing. It has been determined that the amount of radioactive ${}^{14}C$ lost 90% of the amount present at the time of death. Assume the half-life of ${}^{14}C$ is 5,730 years and let $A(t) = A_0\,e^{-k\,t}$ be the amount of ${}^{14}C$ in the specimen after t years.

(a) Use the information about the half-life and determine the exact value of k.

Solution:

The following holds $\frac{A_0}{2} = A(5730) = A_0\,e^{-k \cdot 5730}$.

Conclude that $\ln\left(\frac{1}{2}\right) = -k \cdot 5730$ or $k = \frac{\ln(2)}{5730}$.

(b) Is it necessary to know A_0 to determine the age?

Solution:

No, A_0 appears on both sides and cancels.

(c) Determine the exact value of the age of the specimen.

Solution:

Find t so that $\frac{A_0}{10} = A_0\,e^{-k \cdot t}$. This time $\ln\left(\frac{1}{10}\right) = -k \cdot t$ so $t = \frac{\ln(10)}{\ln(2)} \cdot 5730$.

(d) Give a numerical approximation of the age of the specimen rounded to the nearest thousand years. Also indicate why this may exaggerate the accuracy available based on the data of the problem.

Solution:

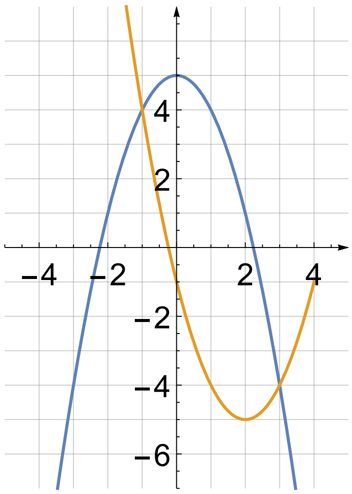
$\frac{\ln(10)}{\ln(2)} \cdot 5730 \approx 19\,035$ so 19,000 years old.

It is not quite clear if the 90% value has 2 accurate digits.

8. Determine the area of the bounded region determined by the curves $f(x) = 5 - x^2$ and $g(x) = x^2 - 4\,x - 1$.

To get full credit sketch the region and also show clearly how all the data you use in the calculation is obtained.

Solution:



Solve $5 - x^2 = x^2 - 4\,x - 1$ or $2\,x^2 - 4\,x - 6 = 2\,(x^2 - 2\,x - 3) = 2\,(x - 3)\,(x + 1) = 0$.

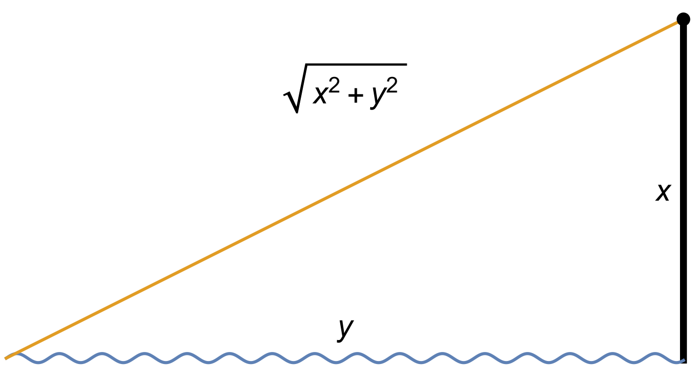
Observe that $5 - x^2$ is the upper curve.

$\int_1^3 ((5 - x^2) - (x^2 - 4\,x - 1))\,dx =$

$2\int_1^3 (3 + 2\,x\, - x^2)\,dx = 2\left[3\,x + x^2 - \frac{x^3}{3}\right]_1^3 = 2\left((9 + 9 - 9) - \left(3 + 1 - \frac{1}{3}\right)\right) = \frac{32}{2}$

9. A large plot of land is situated along a straight section of a river. There is enough money to put up a 2 mile long fence and a single anchor post. The fence will run from the anchor post in two directions. One direction will be perpendicular to the river, and the other will be at some other angle. There will be no fence along the river. It follows that the enclosed region forms a right-angle triangle. Assume the side perpendicular to the river has length x and the side along the river has length y. The remaining side will be the hypotenuse of the triangle. With these constraints determine the configuration that encloses the largest area.

Hint: It is more expedient to maximize the square of the area!



Solution:

The area is given by $\frac{x \cdot y}{2}$.

Following the hint, maximize $\frac{1}{4} \cdot x^2\, y^2$.

The constraint is given by $x + \sqrt{x^2 + y^2} = 2$.

It is easiest to eliminate y using the constraint.

$\sqrt{x^2 + y^2} = 2 - x$ implies that $x^2 + y^2 = (2 - x)^2$ and $y^2 = (2 - x)^2 - x^2 = 4 - 4\,x$.

Maximize $\frac{1}{4} \cdot x^2 (4 - 4\,x) = x^2 - x^3$.

The domain is $0 \leq x \leq 1$ and the area is 0 both when $x = 0$ and when $x = 1$.

It follows that the area must be maximized when the derivative is 0.

The derivative is zero when $2\,x - 3\,x^2 = x(2 - 3\,x) = 0$.

Only $x = \frac{2}{3}$ is of interest.

The corresponding y is $y = \sqrt{4 - 4 \cdot \frac{2}{3}} = \frac{2}{\sqrt{3}}$ so the largest possible area in this case is $\frac{2}{3\sqrt{3}} \approx 0.385$ square miles. Any other configuartion will enclose less area!