March 14, 2014

General Picard Integrator Documentation

1 Description

This gives some documentation corresponding to the code `gen.purpose.picard.int.fox` given in §5. An algorithm is given and then an example is explained and analyzed.

We will consider the initial value problem find $x(t) \in \mathbb{R}^n$ when $x'(t) = F(x)$ and $x(0) = x^0$.

2 Algorithm

Algorithm 1 General Picard Iteration Based Method

input Dimension of ODE, Order of Picard Integration, Number of Time Steps to Do, Time Step Size, Initial Conditions x^0,
output $x(t)$ for $t = J \times$ Time Step Size, $J = 1, 2, \cdots$ Number of Time Steps to Do
READDIMORDNITR; ▷ Reads all input except initial conditions
DIMENSION ← Dimension of ODE
MAXORDER ← Order of Picard Integration
NITR ← Number of Time Steps to Do
DELT(1) ← Time Step Size
END READDIMORDNITR;
$X(I) \leftarrow x_I^0$ Read from imput for $I = 1,\ 2,\ \cdots$,DIMENSION
for $J = 1 \rightarrow$ number of time steps **do**
    $PHI(I) \leftarrow X(I)$ for $I = 1,\ 2,\ \cdots$, DIMENSION
    for $L = 1 \rightarrow$ MAXORDER **do**
        Set DA order to L
        DERIVATIVES;
        PHI2(I) ← $F_I\Big($PHI(1), PHI(2), $\cdots$, PHI(DIMENSION)$\Big)$ for $I = 1, 2 \cdots$, DIMENSION
        END DERIVATIVES;
        PHI2(I) ← $X(I) + \int_0^1 PHI2(I)dt$ for $I = 1, 2 \cdots$, DIMENSION
        PHI(I) ← PHI2(I) for $I = 1, 2 \cdots$, DIMENSION
    end for
    X(I) ← PHI2(I)Evaluated at DELT(1) for $I = 1, 2 \cdots$, DIMENSION
    Write Current Time J × DELT(1) and X to data file
end for

3 Example

For an example we will use the function $y = f(t) = e^{3t}\big(\sin(2t) + \cos(2t)\big)$. We can then compute

$$\begin{aligned} f'(t) &= 3e^{3t}\big(\sin(2t) + \cos(2t)\big) + 2e^{3t}\big(\cos(2t) - \sin(2t)\big) \\ &= e^{3t}\big(\sin(2t) + 5\cos(2t)\big) \\ f''(t) &= 3e^{3t}\big(\sin(2t) + 5\cos(2t)\big) + 2e^{3t}\big(\cos(2t) - 5\sin(2t)\big) \\ &= e^{3t}\big(-7\sin(2t) + 17\cos(2t)\big) \end{aligned}$$

We then can make the ODE below from the known solution.

$$\begin{aligned} &x_1 = e^{3t} && x_1' = 3e^t = 3x_1 && x_1(0) = 1 \\ &x_2 = \cos(2t) && x_2' = -2\sin(2t) = -2x_3 && x_2(0) = 1 \\ &x_3 = \sin(2t) && x_3' = 2\cos(2t) = 2x_2 && x_3(0) = 0 \\ &x_4 = y && x_4' = f'(t) = x_5 && x_4(0) = f(0) = 1 \\ &x_5 = y' && x_5' = f''(t) = x_1(-7x_3 + 17x_2) && x_5(0) = f'(0) = 5 \end{aligned}$$

Since this ODE has derivative components with continuous spacial partial derivatives in all neighborhood of the initial conditions, the ODE has a unique solution. The code was used to solve this problem. First, an expansion about $t = 0$ was done as shown in §4. Then, several iterations at different orders.

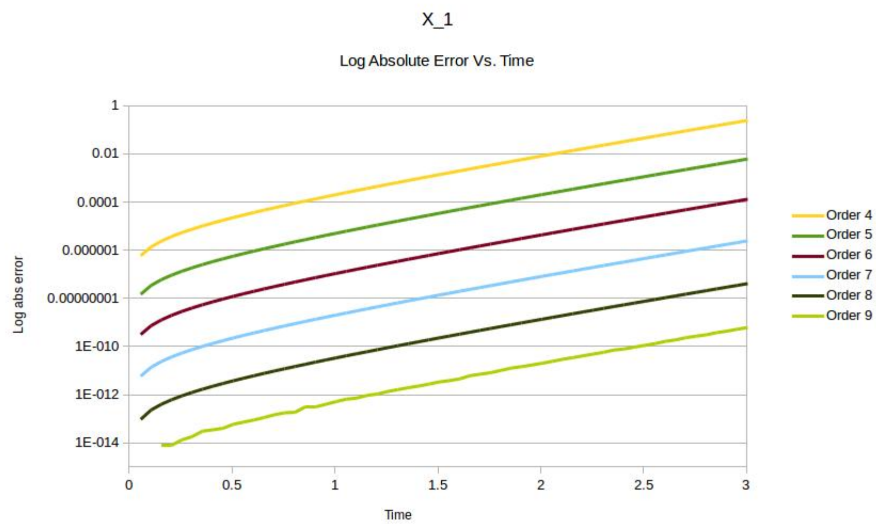
4 Results

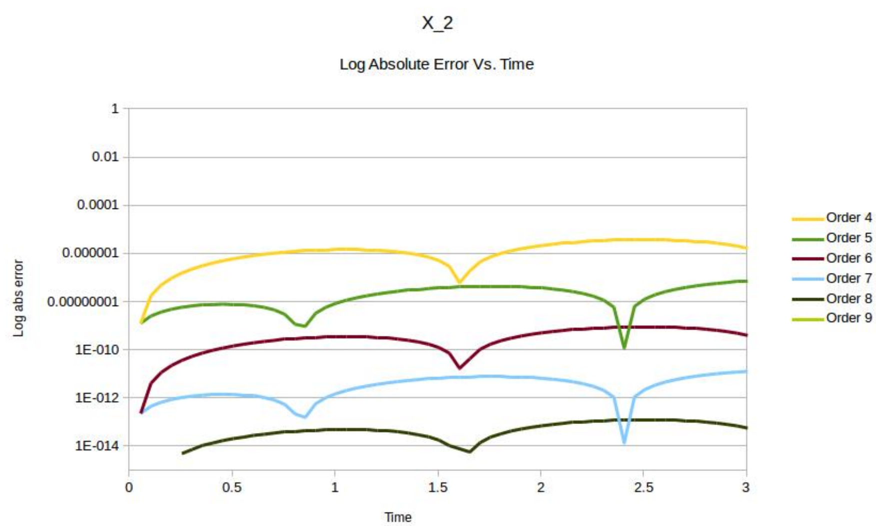
The computed and actual Taylor series of the example closely agree as is shown below. The left column is the computed solution and the right is the expansion of the exact solution about 0 (computed by COSY).

```
gen.purpose.picard.int.example
 --- BEGINNING COMPILATION
 --- BEGINNING EXECUTION
DIMENSION:   5.000000000000000
MAXORDER:   10.00000000000000
NUMBER OF ITERATIONS:   1.000000000000000
STEP SIZE:  0.5000000000000000E-001
COMPUTED SOLUTION
     I  COEFFICIENT           ORDER EXPONENTS
     1   1.000000000000000       0   0
     2   3.000000000000000       1   1
     3   4.500000000000000       2   2
     4   4.500000000000000       3   3
     5   3.375000000000000       4   4
     6   2.025000000000000       5   5
     7   1.012500000000000       6   6
     8  0.4339285714285714       7   7
     9  0.1627232142857143       8   8
    10  0.5424107142857142E-01   9   9
    11  0.1627232142857143E-01  10  10
    ---------------------------------
     I  COEFFICIENT           ORDER EXPONENTS
     1   1.000000000000000       0   0
     2  -2.000000000000000       2   2
     3  0.6666666666666666       4   4
     4  -.8888888888888889E-01   6   6
     5  0.6349206349206349E-02   8   8
     6  -.2821869488536156E-03  10  10
     --------------------------------
     I  COEFFICIENT           ORDER EXPONENTS
     1   2.000000000000000       1   1
     2  -1.333333333333333       3   3
     3  0.2666666666666667       5   5
     4  -.2539682539682540E-01   7   7
     5  0.1410934744268078E-02   9   9
     --------------------------------
     I  COEFFICIENT           ORDER EXPONENTS
     1   1.000000000000000       0   0
     2   5.000000000000000       1   1
     3   8.500000000000000       2   2
     4   6.166666666666667       3   3
     5  0.4166666666666666E-01   4   4
     6  -3.958333333333334       5   5
     7  -3.976388888888890       6   6
     8  -2.183134920634921       7   7
     9  -.7142609126984126       8   8
    10  -.8199680335097000E-01   9   9
    11  0.5397293871252200E-01  10  10
    ---------------------------------
     I  COEFFICIENT           ORDER EXPONENTS
     1   5.000000000000000       0   0
     2   17.00000000000000       1   1
     3   18.50000000000000       2   2
     4  0.1666666666666667       3   3
     5  -19.79166666666667       4   4
     6  -23.85833333333334       5   5
     7  -15.28194444444445       6   6
     8  -5.714087301587301       7   7
     9  -.7379712301587300       8   8
    10  0.5397293871252200       9   9
    11  0.4304334766313928      10  10
    ---------------------------------
```

```
EXACT SOLUTION
     I  COEFFICIENT           ORDER EXPONENTS
     1   1.000000000000000       0   0
     2   3.000000000000000       1   1
     3   4.500000000000000       2   2
     4   4.500000000000000       3   3
     5   3.375000000000000       4   4
     6   2.025000000000000       5   5
     7   1.012500000000000       6   6
     8  0.4339285714285714       7   7
     9  0.1627232142857143       8   8
    10  0.5424107142857142E-01   9   9
    11  0.1627232142857143E-01  10  10
    ---------------------------------
     I  COEFFICIENT           ORDER EXPONENTS
     1   1.000000000000000       0   0
     2  -2.000000000000000       2   2
     3  0.6666666666666666       4   4
     4  -.8888888888888888E-01   6   6
     5  0.6349206349206348E-02   8   8
     6  -.2821869488536155E-03  10  10
     --------------------------------
     I  COEFFICIENT           ORDER EXPONENTS
     1   2.000000000000000       1   1
     2  -1.333333333333333       3   3
     3  0.2666666666666667       5   5
     4  -.2539682539682539E-01   7   7
     5  0.1410934744268077E-02   9   9
     --------------------------------
     I  COEFFICIENT           ORDER EXPONENTS
     1   1.000000000000000       0   0
     2   5.000000000000000       1   1
     3   8.500000000000000       2   2
     4   6.166666666666667       3   3
     5  0.4166666666666607E-01   4   4
     6  -3.958333333333334       5   5
     7  -3.976388888888887       6   6
     8  -2.183134920634921       7   7
     9  -.7142609126984133       8   8
    10  -.8199680335097004E-01   9   9
    11  0.5397293871252228E-01  10  10
    ---------------------------------
     I  COEFFICIENT           ORDER EXPONENTS
     1   5.000000000000000       0   0
     2   17.00000000000000       1   1
     3   18.50000000000000       2   2
     4  0.1666666666666679       3   3
     5  -19.79166666666666       4   4
     6  -23.85833333333333       5   5
     7  -15.28194444444444       6   6
     8  -5.714087301587306       7   7
     9  -.7379712301587346       8   8
    10  0.5397293871252216       9   9
    11  0.4304334766313941      10  10
    ---------------------------------
 === COSY RUN FINISHED ===
```

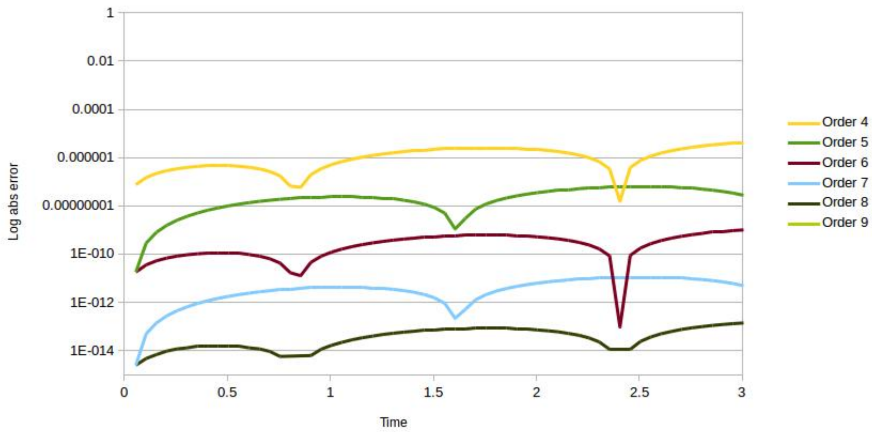
With a time step size of 0.05 and doing 60 iterations to solve the example problem. The following plots were made based on the exact solution. There was no difference between the computed solution from the algorithm and the computed exact solution for x_2 and x_3 at order 9.





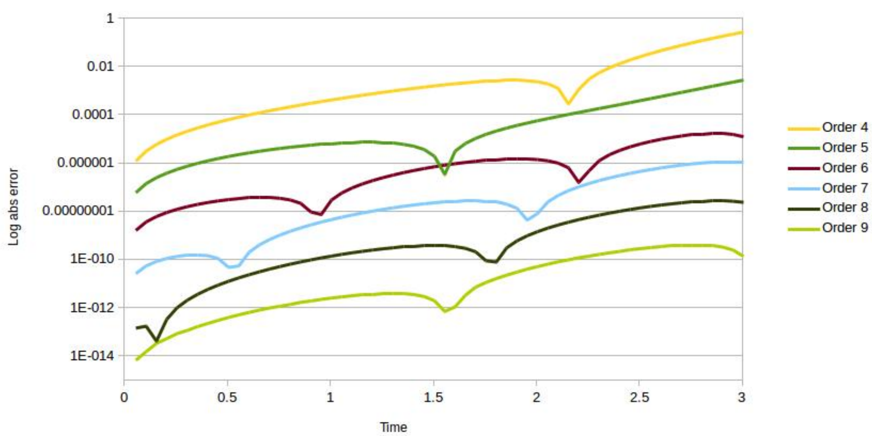
X_3

Log Absolute Error Vs. Time



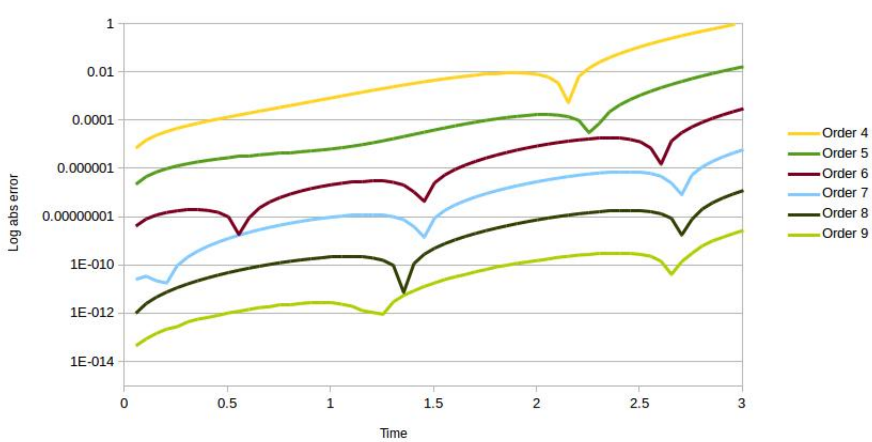
X_4

Log Absolute Error Vs. Time



X_5

Log Absolute Error Vs. Time



5 Code

```
{
General Purpose Picard Iteration Based Integrator
Goto Procedure Derivatives to Change Derivative Function
F(PHI(1),PHI(2), ..., PHI(DIMENSION)).

Input File: gen.purpose.picard.int.input.ssv

Output File: gen.purpose.picard.int.output.ssv

Input File Format
DIMENSION - The dimension of the ODE.
MAXORDER - The number of Picard Iterations to Perform
NITR - The number of iterations
DELT - The step size
Initial Values for x_1 thru x_n on separate lines

Output File Format:

Where t is the elapsed time:

t x_1(t) x_2(t) ... x_DIMENSION(t)
}

BEGIN;

{Global Variables}

VARIABLE DIMENSION 1; {Dimension of ODE}
VARIABLE DUMBY 1; {Dumby Variable}
VARIABLE MAXORDER 1; {Maximum Order For Picard Integrator}
VARIABLE NITR 1; {Maximum Number of Iterations}
VARIABLE DELT 1 1; {TIME STEP SIZE}
VARIABLE STRI 1000; {STRING}

FUNCTION SF R FORM ; RECST R FORM SF ; ENDFUNCTION ; {REAL TO STRING}

PROCEDURE READDIMORDNITR;
OPENF 42 '.\\gen.purpose.picard.int.input.ssv' 'OLD';
READ 42 DIMENSION;
STRI:= 'DIMENSION: ';
STRI:=STRI & ST(DIMENSION);
WRITE 6 STRI;
READ 42 MAXORDER;
STRI:= 'MAXORDER: ';
STRI:=STRI & ST(MAXORDER);
WRITE 6 STRI;
READ 42 NITR;
STRI:= 'NUMBER OF ITERATIONS: ';
STRI:=STRI & ST(NITR);
WRITE 6 STRI;
READ 42 DUMBY;
```

```
DELT(1):=DUMBY;
STRI:='STEP SIZE:   ';
STRI:=STRI&ST(DELT(1));
WRITE 6 STRI;
CLOSEF 42;
ENDPROCEDURE; {READDIMORDNITR}

PROCEDURE PICINT;
VARIABLE X 1 DIMENSION;
VARIABLE I 1;  {Index Variable}
VARIABLE J 1;  {Index Variable}
VARIABLE L 1;  {Index Variable}
VARIABLE PHI (MAXORDER+1) DIMENSION;
VARIABLE PHI2 (MAXORDER+1) DIMENSION;
VARIABLE NM 1;

PROCEDURE DERIVATIVES;
PHI2(1):=3*PHI(1);
PHI2(2):=-2*PHI(3);
PHI2(3):=2*PHI(2);
PHI2(4):=PHI(5);
PHI2(5):=PHI(1)*(-7*PHI(3)+17*PHI(2));
ENDPROCEDURE;  {DERIVATIVES}

DAINI MAXORDER 1 0 NM;
{READING INITIAL CONDITIONS}
OPENF 42 '.\\gen.purpose.picard.int.input.ssv' 'OLD';
READ 42 DUMBY;  {Skipping Dimension}
READ 42 DUMBY;  {Skipping Maximum Order}
READ 42 DUMBY;  {Skipping NITR}
READ 42 DUMBY;  {Skipping DELT}
LOOP I 1 DIMENSION;
  READ 42 DUMBY;
  X(I):=DUMBY;
ENDLOOP;
CLOSEF 42;

STRI:=SF(0,'(E22.16)')& ' ';
LOOP I 1 DIMENSION;
  STRI:=STRI & SF(X(I),'(E22.16)')& ' ';
ENDLOOP;
OPENF 43 '.\\gen.purpose.picard.int.output.ssv' 'REPLACE';
WRITE 43 STRI;

LOOP J 1 NITR;
  {Starting "zeroth" Picard Iteration}
  LOOP I 1 DIMENSION;
    PHI(I):=X(I);
  ENDLOOP;
  {Finished "zeroth" Picard Iteration}
  LOOP L 1 MAXORDER;
    DANOT L;  {LOWERS DA VECTOR ORDER}
    DERIVATIVES;
    LOOP I 1 DIMENSION;
```

```
        PHI2(I):=X(I)+PHI2(I)%(-1);
      ENDLOOP;
      LOOP I 1 DIMENSION;
        PHI(I):=PHI2(I);
      ENDLOOP;
    ENDLOOP;
    {Finished Picard Iteration}
    {Evaluating for the approximate solution}
    POLVAL 1 PHI2 DIMENSION DELT 1 X DIMENSION;

    STRI:=SF(DELT(1)*J,'(E22.16)')& ' ';
    LOOP I 1 DIMENSION;
      STRI:=STRI & SF(X(I),'(E22.16)')& ' ';
    ENDLOOP;
    WRITE 43 STRI;
ENDLOOP;
CLOSEF 43;
ENDPROCEDURE;   {PICINT}

{MAIN EXECUTION:}

READDIMORDNITR;
PICINT;
END;
```