Math 230 Spring 2012

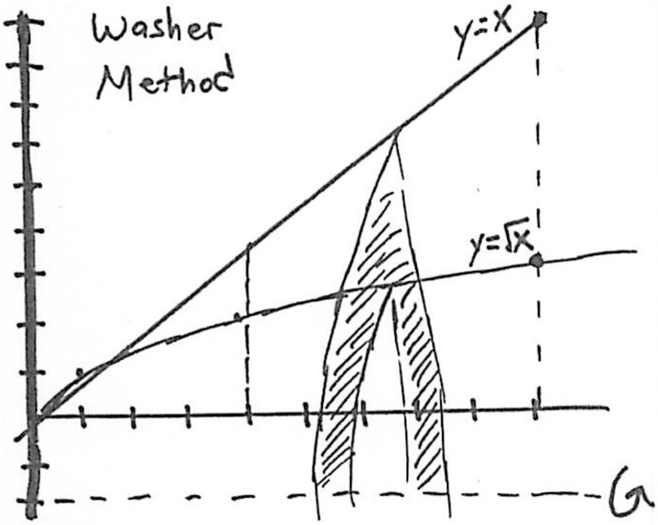
Final Exam

Name: ______________________________

Directions: Please clearly print your name above. There are 7 pages, including this cover sheet. There are 12 problems (some with multiple parts) for a total of 200 points; the point value for each individual problem is clearly marked. You must show your work to receive full credit. Scratch paper is not allowed. You may use the back of the page if you require more space to work a problem.

1.	__________
2.	__________
3.	__________
4.	__________
5.	__________
6.	__________
7.	__________
8.	__________
9.	__________
10.	__________
11.	__________
12.	__________
total:	__________

1 (16 points). Set up, but do not evaluate, an integral for computing the volume of the solid obtained by rotating the region bounded by $y = \sqrt{x}$, $y = x$, $x = 4$, and $x = 9$ about the line $y = -2$. State which method you use to get your integral, and sketch a typical washer or shell, as appropriate.



Outside Radius: $R(x) = x + 2$

Inside Radius: $r(x) = \sqrt{x} + 2$

$$V = \int_4^9 \pi (R(x))^2 - \pi (r(x))^2 dx$$

$$= \int_4^9 \pi (x+2)^2 - \pi (\sqrt{x}+1)^2 dx$$

2 (18 points). Find dy/dx for the following functions, showing all steps.

a) $y = 2^x \ln(1 + x^2)$ Product Rule

$$\frac{dy}{dx} = 2^x \cdot \frac{1}{1+x^2} \cdot 2x + 2^x \ln 2 \cdot \ln(1+x^2)$$

b) $y = \dfrac{\sin^{-1} x}{\tan^{-1} x + e^x}$ Quotient Rule

$\frac{dy}{dx} =$ [redacted] sorry

$$= \frac{(\tan^{-1}x + e^x) \cdot \frac{1}{\sqrt{1-x^2}} - \sin^{-1}(x) \cdot \left(\frac{1}{1+x^2} + e^x\right)}{(\tan^{-1}x + e^x)^2}$$

3 (14 points). Show that the function $f(x) = x^3 + x$ is one-to-one, then find the slope of the tangent line to the graph of $y = f^{-1}(x)$ at the point where $x = 2$.

- you can show $f(x) = x^3+x$ is one-to-one if you show $f'(x) \geq 0$ for all x or $f'(x) \leq 0$ for all x. Note: $f(x)$ is cont. on $(-\infty, \infty)$
- $f'(x) = 3x^2+1$ (has no crit. values) $\overset{+}{\longleftrightarrow}$

  shows $f'(x) > 0$ over $(-\infty, \infty)$.
- slope $= (f^{-1})'(2) = \dfrac{1}{f'(f^{-1}(2))}$

  (1) $f^{-1}(2) = x \rightarrow f(x) = 2$

  $x^3 + x = 2$

  $x = 1$

  so $f^{-1}(2) = 1$

  (2) $f'(x) = 3x^2+1$,

  $= \dfrac{1}{f'(1)}$

  $= \dfrac{1}{4}$

4 (20 points). Evaluate the following limits.

a) $\displaystyle\lim_{x\to\infty} \frac{x^4}{e^x - 1} \overset{LH}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{4x^3}{e^x} \overset{LH}{=} \lim_{x\to\infty} \frac{12x^2}{e^x} \overset{LH}{=} \lim_{x\to\infty} \frac{24x}{e^x}$

$\displaystyle\overset{LH}{=} \lim_{x\to\infty} \frac{24}{e^x} = 0$

b) $\displaystyle\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^{-2x}$ Rewrite as $e^{-2x\ln(1+\frac{1}{x})}$

$\displaystyle\lim_{x\to\infty} -2x\ln\left(1+\frac{1}{x}\right) \overset{-\infty\cdot 0}{=} \lim_{x\to\infty} \frac{-2\ln(1+\frac{1}{x})}{\frac{1}{x}} \overset{LH}{=} \lim_{x\to\infty} \frac{-2\cdot\frac{1}{1+\frac{1}{x}}\cdot\frac{-1}{x^2}}{\frac{-1}{x^2}}$

$\displaystyle= \lim_{x\to\infty} \frac{-2}{1+\frac{1}{x}} = \frac{-2}{1+0} = -2$

Answer: e^{-2}

Also note: $\displaystyle\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$ and $\displaystyle\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{-2x} = \lim_{x\to\infty}\left[\left(1+\frac{1}{x}\right)^x\right]^{-2} = e^{-2}$

5 (24 points). Evaluate the following integrals.

a) $\int_0^1 x^3\sqrt{1-x^2}\,dx$ Trig sub

(1) let $x = \sin\theta$, $dx = \cos\theta\, d\theta$

(2) Bounds: $1 = \sin\theta \to \theta = \pi/2$

$0 = \sin\theta \to \theta = 0$

(3) $\int_0^{\pi/2} \sin^3\theta \sqrt{1-\sin^2\theta} \cdot \cos\theta\, d\theta$

$\to \int_0^{\pi/2} \sin^3\theta \cos^2\theta\, d\theta = \int_0^{\pi/2} \sin^2\theta \cos^2\theta \cdot \sin\theta\, d\theta$

Let $u = \cos\theta$, $du = -\sin\theta\, d\theta \to -du = \sin\theta\, d\theta$

$\int_1^0 -(1-u^2)u^2\,du = \int_1^0 -u^2 + u^4\,du$

$= -\frac{1}{3}u^3 + \frac{1}{5}u^5 \Big|_1^0 = [0+0] - [-\frac{1}{3} + \frac{1}{5}] = \frac{2}{15}$

b) $\int x^2 e^x\,dx$ By Parts

$u = x^2$, $dv = e^x dx$

$du = 2x\,dx$ $v = e^x$

$uv - \int v\,du$

$= x^2 e^x - \int 2xe^x dx$

$x^2 e^x - \int 2xe^x dx$

$-\left[\begin{array}{ll} u = 2x & dv = e^x dx \\ du = 2dx & v = e^x \end{array}\right]$

$= x^2 e^x - \left[2xe^x - \int 2e^x dx\right]$

$= x^2 e^x - 2xe^x + 2e^x dx \quad + C$

6 (12 points). Determine whether the improper integral $\int_0^\infty \frac{e^{-x}}{x+1}\,dx$ converges or diverges. State your reasoning.

$\frac{e^{-x}}{(x+1)} = \frac{1}{e^x(x+1)} < \frac{1}{e^x}$ on $(0,\infty)$

So $\int_0^\infty \frac{e^{-x}}{x+1}\,dx < \int_0^\infty \frac{1}{e^x}\,dx = \int_0^\infty e^{-x}dx = -e^{-x}\Big|_0^\infty = 1$

Since $0 < \int_0^\infty \frac{e^{-x}}{x+1}\,dx < \int_0^\infty e^{-x}dx$ and $\int_0^\infty e^{-x}dx$ converges

then $\int_0^\infty \frac{e^{-x}}{x+1}\,dx$ converges by DCT

7 (16 points). Find a polynomial that approximates e^{-x} to within 10^{-4} for all x in the interval $[0,1]$. Explain why your polynomial has the required accuracy.

8 (16 points). Suppose $\sum_{n=0}^{\infty} c_n(x-1)^n$ converges when $x=-4$ and diverges when $x=8$. What, if anything, can you say about the following series? Be sure to state your reasoning.

a) $\displaystyle\sum_{n=0}^{\infty} c_n$

b) $\displaystyle\sum_{n=0}^{\infty} c_n(-1)^n 9^n$

c) $\displaystyle\sum_{n=0}^{\infty} c_n(-1)^n 6^n$

d) $\displaystyle\sum_{n=0}^{\infty} n c_n 3^{n-1}$

9 (10 points). What does it mean to say a series converges absolutely?

Given $\sum a_n$, $\sum a_n$ is said to converge absolutely if $\sum |a_n|$ converges.

10 (20 points). Determine whether the following series converge absolutely, converge conditionally, or diverge. Be sure to state your reasoning.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{2n^3+2}$

$b_n = \frac{n^2}{2n^3+2}$

(1) $b_n > 0$ ✓
(2) b_n decreases ✓
(3) $\lim_{n\to\infty} b_n = 0$ ✓

$\Rightarrow \sum \frac{(-1)^n n^2}{2n^3+2}$ converges by AST

Need to check for absolute convergence

$\sum \left| \frac{(-1)^n n^2}{2n^3+2} \right| = \sum \frac{n^2}{2n^3+2}$ and $\sum \frac{n^2}{2n^3+2} \approx \sum \frac{n^2}{2n^3} \approx \sum \frac{1}{n}$ ← diverges

LCT: $\lim_{n\to\infty} \frac{n^2}{2n^3+2} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n^3}{2n^3+2} = \frac{1}{2}$ (finite #)

Since $\sum \frac{1}{n}$ diverges (p-test), $\sum \frac{n^2}{2n^3+2}$ diverges by LCT.

Therefore, $\sum \frac{(-1)^n n^2}{2n^3+2}$ converges conditionally

b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 100^n}{n!}$

Ratio Test

$a_{n+1} = \frac{(-1)^{n+1} 100^{n+1}}{(n+1)!}$, $a_n = \frac{(-1)^n 100^n}{n!}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} 100^{n+1}}{(n+1)!} \cdot \frac{n!}{(-1)^n 100^n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(-1)^n} \cdot \frac{100^{n+1}}{100^n} \cdot \frac{n!}{(n+1)!} \right|$$

$$= \lim_{n\to\infty} \left| (-1) \cdot 100 \frac{1}{n+1} \right| = 1 \cdot 100 \cdot 0 = 0 \ \underline{< 1}$$

By Ratio Test, $\sum \frac{(-1)^n 100^n}{n!}$ converges

11 (24 points). Find power series for the following functions, and state the radii of convergence.

a) $\ln(1+x^2)$

$$\frac{d}{dx}\left[\ln(1+x^2)\right] = \frac{2x}{1+x^2} = 2x \cdot \frac{1}{1-(-x^2)} = 2x \sum_{n=0} (-x^2)^n = 2x \sum_{n=0} (-1)^n x^{2n} = \sum 2(-1)^n x^{2n+1}$$

Integrate to get back $\ln(1+x^2) = \int \sum 2(-1)^n x^{2n+1}\,dx$

$$= \sum 2(-1)^n \frac{x^{2n+2}}{2n+2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n+1}$$

b) $\dfrac{1-\cos x}{x^2}$

$$= \frac{1-\sum \frac{(-1)^n x^{2n}}{(2n)!}}{x^2} = \frac{1-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}\cdots\right)}{x^2} = \frac{\frac{x^2}{2!}-\frac{x^4}{4!}+\frac{x^6}{6!}-\frac{x^8}{8!}+\cdots}{x^2}$$

$$= \frac{1}{2!}-\frac{x^2}{4!}+\frac{x^4}{6!}-\frac{x^6}{8!}+\cdots = \sum_{n=0} \frac{(-1)^n x^{2n}}{(2n+2)!}$$

c) $\dfrac{1}{(1-x)^2}$

12 (10 points). The graph $y = f(x)$ of a function is shown. The integral $\int_0^4 f(x)\,dx$ is estimated by the midpoint and trapezoid rules using ten subintervals. Which is the larger estimate, and are these estimates greater than or less than the actual integral? State your reasoning.