MATH 230 FINAL EXAM (Sping 2011)

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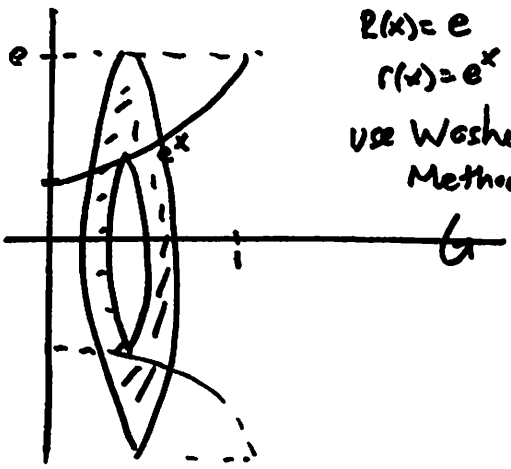
20 questions, 10 points for each question. Show all work for full credit.

1. Find the volume of the solid generated when the region bounded by $y = e^x$, $y = e$, and $x = 0$ is rotated about the x-axis. Use washer method.



$R(x) = e$

$r(x) = e^x$

use Washer Method

$$\int_0^1 \pi(e)^2 - \pi(e^x)^2\,dx$$

$$= \int_0^1 \pi e^2 - \pi e^{2x}\,dx$$

$$= \pi e^2 x - \frac{\pi}{2}e^{2x}\Big|_0^1$$

$$= \left[\pi e^2 - \frac{\pi}{2}e^2\right] - \left[0 - \frac{\pi}{2}\right]$$

$$= \frac{\pi}{2}e^2 + \frac{\pi}{2} \text{ or } \frac{\pi}{2}(e^2+1)$$

2. Evaluate $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$.

$$\overset{LH}{\underset{\infty/\infty}{=}} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$$

3. Find the derivative of $y = \tan^{-1}(e^x) + e^{\sin^{-1} x}$.

$$y' = \frac{1}{1+(e^x)^2}\cdot e^x + e^{\sin^{-1}x}\cdot\frac{1}{\sqrt{1-x^2}}$$

$$= \frac{e^x}{1+e^{2x}} + \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$$

4. Find the derivative of $y = (1+x^2)^x$. $= e^{x\ln(1+x^2)}$. Note: $a^u = e^{u\ln a}$

$$y' = e^{x\ln(1+x^2)} \cdot \frac{d}{dx}\left[x\ln(1+x^2)\right]$$

$$= e^{x\ln(1+x^2)} \cdot \left(x \cdot \frac{1}{1+x^2} \cdot 2x + 1 \cdot \ln(1+x^2)\right)$$

$$= e^{x\ln(1+x^2)} \cdot \left(\frac{2x^2}{1+x^2} + \ln(1+x^2)\right)$$

5. The half-life of Radium-226 is 1590 years. Suppose you have a 100-mg sample. When will the mass be reduced to 15-mg?

6. Evaluate $\int \cos^4 x \sin^3 x dx$. $= \int \cos^4 x \sin^2 x \cdot \sin x \, dx$

(1) Let $u = \cos x$, $du = -\sin x \, dx$ $\longrightarrow$ $\int -u^4(1-u^2)\,du$

$* \sin^2 x = 1 - \cos^2 x = 1 - u^2$

(2) $\int -u^4 + u^6 \, du = -\frac{1}{5}u^5 + \frac{1}{7}u^7 + C$

$$= \frac{-1}{5}(\cos x)^5 + \frac{1}{7}(\cos x)^7 + C$$

7. Evaluate $\int x \ln x dx$. By Parts

$u = \ln x \quad dv = x\,dx$

$du = \frac{1}{x}dx \quad v = \frac{x^2}{2}$

$uv - \int v\,du$

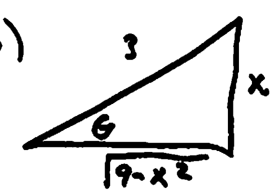
$\frac{x^2}{2}\ln x - \int \frac{x}{2}dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$

8. Evaluate $\int \frac{1}{x^2\sqrt{9-x^2}}dx$. Trig Sub

(1) $x = 3\sin\theta,\ dx = 3\cos\theta\,d\theta,\ x^2 = 9\sin^2\theta$

(2) $\int \frac{3\cos\theta\,d\theta}{9\sin^2\theta\sqrt{9-(3\sin\theta)^2}} = \int \frac{3\cos\theta\,d\theta}{9\sin^2\theta \cdot 3\cos\theta} = \int \frac{1}{9\sin^2\theta}d\theta = \int \frac{1}{9}\csc^2\theta\,d\theta$

$= -\frac{1}{9}\cot\theta + C$

(3) 

$-\frac{1}{9}\cot\theta + C = -\frac{1}{9}\left(\frac{\sqrt{9-x^2}}{x}\right) + C$

9. Evaluate $\int \frac{x-9}{(x+5)(x-2)}dx$.

(1) $\frac{x-9}{(x+5)(x-2)} = \frac{A}{x+5} + \frac{B}{x-2} \longrightarrow x-9 = A(x-2) + B(x+5)$

(2) $x=2: \quad -7 = 7B \rightarrow B = -1$

$x=-5: \quad -14 = -7A \rightarrow A = 2$

(3) $\int \frac{2}{x+5} + \frac{-1}{x-2}dx = 2\ln|x+5| - \ln|x-2| + C$

10. Write the definition and evaluate the improper integral $\int_1^\infty xe^{-x^2}\,dx$ if it is convergent.

$\lim_{t\to\infty}\int_1^t xe^{-x^2}dx$ $\hookrightarrow$ or $\rightarrow$ $\left\{\begin{array}{l} \text{Let } u=-x^2 \\ du=-2x\,dx \rightarrow -\frac{1}{2}du = x\,dx \\ \text{If } x=\infty,\ u=-\infty^2=-\infty \\ \text{If } x=1,\ u=-1 \end{array}\right.$

$$\int_{-1}^{-\infty}-\frac{1}{2}e^u\,du = \lim_{b\to-\infty}\int_{-1}^{b}-\frac{1}{2}e^u\,du$$

$$\lim_{b\to-\infty}-\frac{1}{2}e^u\Big|_{-1}^{b} = \lim_{b\to-\infty}\left[-\frac{1}{2}e^b+\frac{1}{2}e^{-1}\right]$$

$$= \frac{1}{2}e^{-1}$$

convergent

11. Find the length of $y=\ln(\cos x)$, $0\le x\le \pi/3$.

(1) Arc length: $\int_0^{\pi/3}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

(2) $\frac{dy}{dx}=\frac{1}{\cos x}\cdot -\sin x = -\tan x$

$\left(\frac{dy}{dx}\right)^2=\tan^2 x$

(3) $\int_0^{\pi/3}\sqrt{1+\tan^2 x}\,dx = \int_0^{\pi/3}\sqrt{\sec^2 x}\,dx$

$$= \int_0^{\pi/3}\sec x\,dx = \ln\left|\sec x+\tan x\right|\Big|_0^{\pi/3}$$

$$= \left[\ln|2+\sqrt{3}|\right]-\left[\ln|1+0|\right] = \ln|2+\sqrt{3}|$$

12. Given $f(x)=2x^3+5$, show that f is one-to-one and find f^{-1}.

- If f is continuous and $f'(x)>0$ or $f'(x)<0$ for all x then f is one to one.
- $f'(x)=6x^2>0$ for all x. So f is one-to-one.
- Find f^{-1}: (1) $y=2x^3+5$

(2) switch x and y $\rightarrow$ $x=2y^3+5$

(3) Solve for y: $x-5=2y^3$

$\sqrt[3]{\frac{x-5}{2}}=y$ so $f^{-1}(x)=\sqrt[3]{\frac{x-5}{2}}$

13. Determine whether the series $\Sigma_{n=1}^{\infty}\frac{3^n}{n!}$ is convergent or divergent (mention the theorem you use).

Ratio Test

$a_n = \frac{3^n}{n!}$, $a_{n+1} = \frac{3^{n+1}}{(n+1)!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}}{(n+1)!}\cdot\frac{n!}{3^n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}}{3^n}\cdot\frac{n!}{(n+1)n!}\right| = \lim_{n\to\infty}\left|3\cdot\frac{1}{n+1}\right|$$

$$= 3\cdot 0$$

$$= 0 < 1$$

converges by Ratio Test

14. Determine whether the series $\Sigma_{n=1}^{\infty}(1+\frac{1}{n})^n$ is convergent or divergent (mention the theorem you use).

Diverges by divergence test

$\downarrow$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e^1 \neq 0$$

$\uparrow$

Memorize this

15. Show that the series $\Sigma_{n=1}^{\infty}\frac{(-1)^n 2^n}{3^{n+1}}$ is a convergent geometric series and find the sum.

$$\sum_{n=0}\frac{(-1)^{n+1}2^{n+1}}{3^{n+2}} = \sum_{0}\frac{(-1)(-1)^n\cdot 2\cdot 2^n}{3^2\cdot 3^n} = \sum_{0}\frac{-2}{3^2}\left(\frac{-2}{3}\right)^n$$

$-1 < -\frac{2}{3} < 1$ so its convergent

$$= \frac{a}{1-r} = \frac{-2/3^2}{1-(-2/3)} = \frac{-2/3^2}{5/3} = \frac{-2}{15}$$

16. Determine whether the series $\Sigma_{n=1}^{\infty}(-1)^n\frac{n}{n^2+1}$ is absolutely convergent, conditionally convergent, or divergent (mention the theorems you use).

conditionally convergent

use AST with LCT

(1) $\sum(-1)^n \frac{n}{n^2+1}$ conv. by AST

(2) $\sum_1 \left|\frac{(-1)^n n}{n^2+1}\right| = \sum_1 \frac{n}{n^2+1}$ div. when compared with $\sum \frac{1}{n}$ using LCT

17. Determine whether the series $\Sigma_{n=1}^{\infty} \ln\frac{n}{n+1}$ is convergent by definition.

(1) $\sum_1 \ln(n/n+1) = \sum_1 \ln(n) - \ln(n+1)$

(2) Find a formula for S_n

$S_n = [\ln(1) - \ln(2)] + [\ln 2 - \ln 3] + [\ln 3 - \ln 4] + \cdots + [\ln n - \ln(n+1)]$

$= \ln n - \ln(n+1)$

18. Find the interval of convergence for $\Sigma_{n=1}^{\infty}\frac{(x-3)^n}{\sqrt{n}}$.

use Ratio Test

$[2, 4)$

19. Find the 3rd-degree Taylor polynomial of $\sqrt{2x+1}$ at $a = 0$.

$f^{(n)}(x)$	$f^{(n)}(0)$	$c_n = \frac{f^{n}(0)}{n!}$
$f(x) = \sqrt{2x+1}$	$f(0) = 1$	$c_0 = \frac{1}{0!} = 1$
$f'(x) = \frac{1}{\sqrt{2x+1}}$	$f'(0) = 1$	$c_1 = \frac{1}{1!} = 1$
$f''(x) = \frac{-1}{(2x+1)^{3/2}}$	$f''(0) = -1$	$c_2 = \frac{-1}{2!} = \frac{-1}{2}$
$f'''(x) = \frac{3}{(2x+1)^{5/2}}$	$f'''(0) = 3$	$c_3 = \frac{3}{3!} = \frac{1}{2}$

$$T_3(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$$
$$= 1 + x - \frac{1}{2}x + \frac{1}{2}x^3$$

20. Find the Maclaurin series for $\sin x^2$ and use it to evaluate $\int \sin x^2 dx$.

If $\sin x = \sum_{n=0} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, then $\sin(x^2) = \sum_{n=0} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!}$

$$= \sum \frac{(-1)^n x^{4n+2}}{(2n+1)!}$$

$$\int \sin(x^2)dx = \int \sum \frac{(-1)^n x^{4n+2}}{(2n+1)!} dx$$

$$= \sum_{n=0} \frac{(-1)^n x^{4n+3}}{(4n+3)(2n+1)!}$$