Math 230 Spring 2010

Final Exam

Name: ______________________________

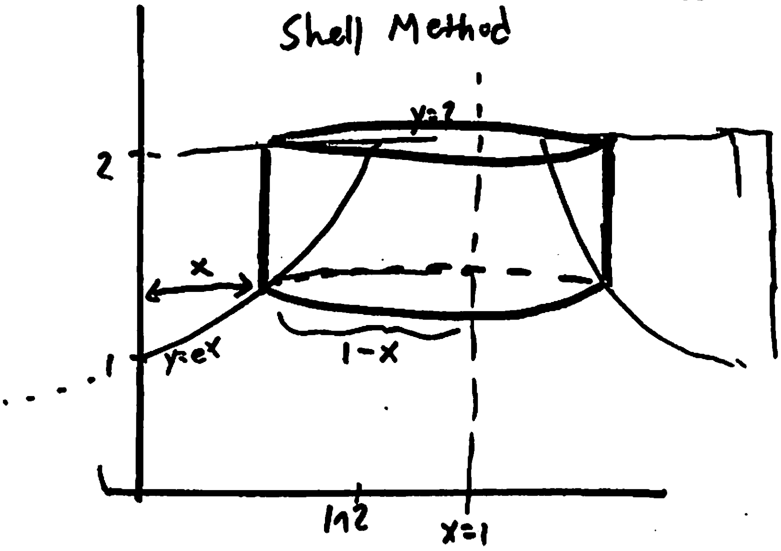
Directions: Please clearly print your name above. There are 8 pages, including this cover sheet. There are 11 problems (some with multiple parts) for a total of 200 points; the point value for each individual problem is clearly marked. You must show your work to receive full credit. Scratch paper is not allowed. You may use the back of the page if you require more space to work a problem.

1. __________
2. __________
3. __________
4. __________
5. __________
6. __________
7. __________
8. __________
9. __________
10. __________
11. __________

total: __________

1 (16 points). Set up, but do not evaluate, an integral for computing the volume of the solid obtained by rotating the region bounded by $y = e^x$, $x = 0$ and $y = 2$ about the line $x = 1$. State which method you use to get your integral, and sketch a typical washer or shell, as appropriate.

Shell Method



$$\int_a^b 2\pi(R)(H)\,dx$$

$$\int_0^{\ln 2} 2\pi(1-x)(2-e^x)\,dx$$

2 (20 points). Find dy/dx for the following functions, showing all steps.

a) $y = e^{x^2} \sin^{-1} x$

$$\frac{dy}{dx} = e^{x^2} \cdot \frac{1}{\sqrt{1-x^2}} + 2x e^{x^2} \cdot \sin^{-1} x$$

b) $y = \dfrac{\tan^{-1} x}{\ln(1+x)}$

$$\frac{dy}{dx} = \frac{\ln(1+x) \cdot \frac{1}{1+x^2} - \tan^{-1} x \cdot \left(\frac{1}{1+x}\right)}{\left[\ln(1+x)\right]^2}$$

3 (14 points). Show that the function $f(x) = x^5 + x - 4$ is one-to-one and find $(f^{-1})'(-2)$.

- If f is continuous and $f'(x) > 0$ for all x (or $f'(x) < 0$) then f is one to one.
- $f'(x) = 5x^4 + 1 > 0$ for all x. So f is one-to-one.
- $(f^{-1})'(-2) = \dfrac{1}{f'(f^{-1}(-2))}$

$= \dfrac{1}{6}$

$\leftarrow \begin{cases} (1)\ f^{-1}(-2) = b \rightarrow f(b) = -2 \\ \qquad \rightarrow b^5 + b - 4 = 0 \\ \qquad \rightarrow b = 1 \\ (2)\ f'(f^{-1}(-2)) = f'(1) = 5(1)^4 + 1 = 6 \end{cases}$

4 (18 points). Evaluate the following limits.

a) $\lim_{x \to 0} \dfrac{1 - \cos x}{e^x - 1} \overset{LH}{\underset{0/0}{=}} \lim_{x \to 0} \dfrac{\sin x}{e^x} = \dfrac{0}{e^0} = \dfrac{0}{1} = \underline{\underline{0}}$

b) $\lim_{x \to \infty} \left(1 - \dfrac{2}{x}\right)^x = \lim_{x \to \infty} e^{x \ln(1 - \frac{2}{x})}$

$\lim_{x \to \infty} \underbrace{x \ln(1 - \tfrac{2}{x})}_{\infty \cdot 0} = \lim_{x \to \infty} \dfrac{\ln(1 - \frac{2}{x})}{\frac{1}{x}} \overset{LH}{\underset{0/0}{=}} \lim_{x \to \infty} \dfrac{\frac{1}{1 - \frac{2}{x}} \cdot \frac{2}{x^2}}{\frac{-1}{x^2}}$

$= \lim_{x \to \infty} \dfrac{-2}{1 - \frac{2}{x}} = -2$

Final Answer: $\lim_{x \to \infty} (1 - \frac{2}{x})^x = e^{-2}$

5 (32 points). Evaluate the following integrals.

(1) a) $\int_2^3 \frac{1}{x^2-1}\,dx = \int_2^3 \frac{1}{(x-1)(x+1)}\,dx = \int_2^3 \frac{A}{x-1} + \frac{B}{x+1}\,dx$

(2) $\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} \Rightarrow 1 = A(x+1) + B(x-1)$

(3) Find A, B

$x=1 \rightarrow 1 = A\cdot 2 + 0 \rightarrow A = \frac{1}{2}$, $x=-1$: $1 = B(-2)$, $B = \frac{-1}{2}$

(4) $\int_2^3 \frac{1/2}{x-1} + \frac{-1/2}{x+1}\,dx$

$= \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| \Big|_2^3$

$= \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| \Big|_2^3$

$= \frac{1}{2}\ln\left|\frac{2}{4}\right| - \frac{1}{2}\ln\left|\frac{1}{3}\right|$

b) $\int_0^\pi e^x \sin x\,dx$

c) $\int_e^\infty \frac{1}{x(\ln x)^2}\,dx$ u-sub

(1) Let $u = \ln x$

$du = \frac{1}{x}dx$

(2) If $x=\infty$, $u = \ln\infty = \infty$

If $x = e$, $u = \ln e = 1$

(3) $\int_1^\infty \frac{1}{u^2}\,du = -\frac{1}{u}\Big|_1^\infty = 1$

Offical:

$\lim_{b\to\infty} -\frac{1}{u}\Big|_1^b = \lim_{b\to\infty} -\frac{1}{b} + \frac{1}{1} = 0 + 1 = 1$

d) $\int \frac{t^3}{\sqrt{t^2+1}}\,dt$ trig sub

(1) Let $t = \tan\theta$

$dt = \sec^2\theta\,d\theta$

$t^3 = \tan^3\theta$

(2) $\int \frac{\tan^3\theta \cdot \sec^2\theta\,d\theta}{\sqrt{\tan^2\theta+1}}$

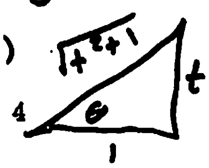
$= \int \tan^3\theta \sec\theta\,d\theta$

(3) $\int \tan^2\theta \sec\theta \tan\theta\,d\theta$

$= \int (\sec^2\theta - 1)\sec\theta\tan\theta\,d\theta$

Let $u = \sec\theta$, $du = \sec\theta\tan\theta\,d\theta$

$= \int u^2 - 1\,du = \frac{1}{3}u^3 - u + C = \frac{1}{3}(\sec\theta)^3 - \sec\theta + C$

(4) 

$\longrightarrow = \frac{1}{3}\left(\sqrt{t^2+1}\right)^3 - \sqrt{t^2+1} + C$

6 (24 points). Determine whether the following series converge absolutely, converge conditionally, or diverge.

a) $\sum_{n=1}^{\infty}(-1)^n \frac{n^2}{n^3+2}$ converge conditionally

(1) $\sum_{1} (-1)^n \frac{n^2}{n^3+2}$ conv. by AST

(2) $\sum_{1} \left| \frac{(-1)^n n^2}{n^3+2} \right| = \sum_{1} \frac{n^2}{n^3+2}$ diverges by LCT with $\sum_{1}^{\infty} \frac{1}{n}$

b) $\sum_{n=1}^{\infty} \frac{n^4}{e^n}$ Ratio Test

$a_{n+1} = \frac{(n+1)^4}{e^{n+1}}$ $a_n = \frac{n^4}{e^n}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^4}{e^{n+1}} \cdot \frac{e^n}{n^4} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^4}{n^4} \cdot \frac{e^n}{e^n \cdot e} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^4}{n^4} \cdot \frac{1}{e} \right| = \frac{1}{e} < 1$$

Since $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$, $\sum \frac{n^4}{e^n}$ conv. by Ratio Test

c) $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}-1}$

converges conditionally

(1) $\sum_{2} \frac{(-1)^{n+1}}{\sqrt{n}-1}$ conv. by AST

(2) $\sum \left| \frac{(-1)^{n+1}}{\sqrt{n}-1} \right| = \sum \frac{1}{\sqrt{n}-1} > \sum \frac{1}{n^{1/2}}$, div. by p-test

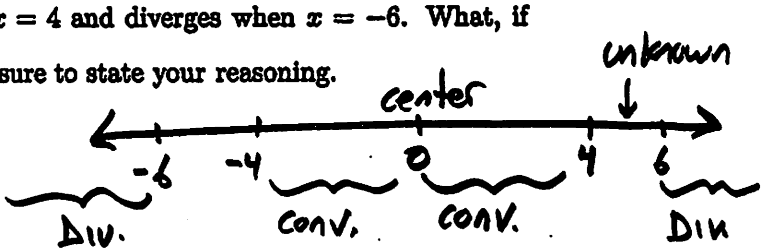
so $\sum \frac{1}{\sqrt{n}-1}$ diverges by DCT.

7 (16 points). Compute the second degree Taylor polynomial $T_2(x)$ for $f(x) = \tan^{-1}(x^2)$ centered at 0.

$f^{(n)}(x)$	$f^{(n)}(0)$	$c_n = \frac{f^{(n)}(0)}{n!}$
$f(x) = \tan^{-1}(x^2)$	$f(0) = 0$	$c_0 = \frac{0}{0!} = 0$
$f'(x) = \frac{2x}{1+x^4}$	$f'(0) = 0$	$c_1 = \frac{0}{1!} = 0$
$\downarrow$ quotient rule		
$f''(x) = \frac{-6x^4+2}{(1+x^4)^2}$	$f''(0) = 2$	$c_2 = \frac{2}{2!} = 1$

$$T_2(x) = c_0 + c_1 x + c_2 x^2$$
$$= 0 + 0x + 1x^2$$
$$= x^2$$

8 (12 points). Suppose $\sum_{n=0}^{\infty} c_n x^n$ converges when $x = 4$ and diverges when $x = -6$. What, if anything, can you say about the following series? Be sure to state your reasoning.



a) $\sum_{n=0}^{\infty} c_n 5^n$

$\uparrow$ $x=5$, which is in a region where convergence is unknown.

b) $\sum_{n=0}^{\infty} c_n(-1)^n 9^n = \sum_{0} c_n (-9)^n$

$\uparrow$ $x=-9$, which is in a region where the series must diverge.

c) $\sum_{n=0}^{\infty} n c_n 3^{n-1}$ $\sum n c_n x^{n-1}$ $\frac{d}{dx}\left[\sum c_n x^n\right]$ so $\sum n c_n x^{n-1}$ must share the same interval of convergence as $\sum c_n x^n$

Since $x=3$ is in a region where $\sum c_n x^n$ converges, then $\sum n c_n x^{n-1}$ also conv. at $x=3$

9 (20 points). Use power series to approximate $\int_0^{1/10} e^{-x^2}\,dx$ with an error less than 10^{-5}. Explain why your estimate is good enough. (Feel free to express your estimate as a sum of fractions.)

10 (14 points). Write out the first 4 terms of the sequence $a_n = 10^n/n!$ and find $\lim_{n\to\infty} a_n$.

$a_0 = \frac{10^0}{0!} = 1$

$a_1 = \frac{10^1}{1!} = 10$

$a_2 = \frac{10^2}{2!} = 50$

$a_3 = \frac{10^3}{3!} = \frac{500}{3} \approx 166.\overline{6}$

$a_4 = \frac{10^4}{4!} = \frac{1250}{3} = 416.67$

• $\lim_{n\to\infty} \frac{10^n}{n!} = 0$

• Use squeeze Thm to prove

$$a_n = \frac{\overbrace{10\cdot 10\cdot 10\cdot 10\cdot \cdots 10\cdot 10\cdots\cdot 10}^{n\text{ times}}}{1\cdot 2\cdot 3\cdot 4\cdots 10\cdot 11\cdots n}$$

$$= \underbrace{\frac{10\cdot 10\cdots 10}{1\cdot 2\cdots 10}}_{\text{call } F}\cdot\underset{<1}{\frac{10}{11}}\cdot\underset{<1}{\frac{10}{12}}\cdots\frac{10}{n}$$

$0 < a_n \leq F\cdot\frac{10}{n}$ *F is a constant

and $\lim_{n\to\infty} \frac{F\cdot 10}{n} = 0$ By Squeeze Thm $\lim_{n\to\infty} a_n = 0$

11 (14 points). Find the radius of convergence of $\sum_{n=0}^{\infty}(-1)^n \frac{n}{4^n}(x+1)^n$.

Start with Ratio Test $a_n = (-1)^n \frac{n}{4^n}(x+1)^n$

$a_{n+1} = (-1)^{n+1} \frac{(n+1)}{4^{n+1}} \cdot (x+1)^{n+1}$

(1) $\lim \left| \frac{a_{n+1}}{a_n} \right| = \lim \left| \frac{(-1)^{n+1}(n+1)(x+1)^{n+1}}{4^{n+1}} \cdot \frac{4^n}{(-1)^n n (x+1)^n} \right|$

$= \lim \left| \frac{(-1)^{n+1}}{(-1)^n} \cdot \frac{n+1}{n} \cdot \frac{4^n}{4^{n+1}} \cdot \frac{(x+1)^{n+1}}{(x+1)^n} \right|$

$= \lim \left| (-1) \cdot \frac{n+1}{n} \cdot \frac{1}{4} \cdot (x+1) \right|$

$= \left| \frac{1}{4}(x+1) \right|$

center: $a = -1$

radius: $R = 4$

(2) Solve $\left| \frac{1}{4}(x+1) \right| < 1$ or $|x+1| < 4$

$-4 < x+1 < 4$

$-5 < x < 3$

(3) Check endpoints:

$x = -5$: $\sum (-1)^n \frac{n}{4^n}(-4)^n = \sum n$ diverges b.c. $\lim_{n\to\infty} n = \infty$

$x = 3$: $\sum (-1)^n \frac{n}{4^n}(4)^n = \sum (-1)^n n$ diverges

Interval of convergence: $(-5, 3)$ or $-5 < x < 3$

Radius of convergence: $R = 4$