Strength of tail dependence based on conditional tail expectation

Lei Hua∗ Harry Joe†

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Abstract. We use the conditional distribution and conditional expectation of one random variable given the other one being large to capture the strength of dependence in the tails of a bivariate random vector. We study the tail behavior of the boundary conditional cumulative distribution function (cdf) and two forms of conditional tail expectation (CTE) for various bivariate copula families. In general, for nonnegative dependence, there are three levels of strength of dependence in the tails according to the tail behavior of CTEs: asymptotically linear, sub-linear and constant. For each of these three levels, we investigate the tail behavior of CTEs for the marginal distributions belonging to maximum domain of attraction of Fréchet and Gumbel, respectively, and for copula families with different tail behavior.

Key words. Tail order, intermediate tail dependence, tail quadrant independence, stochastic increasing, tail behavior, copula, boundary conditional distribution, maximum domain of attraction.

1 Introduction

Tail behavior in terms of strength of dependence in the tails of a copula can have a lot of influence on inferences such as joint tail probability and risk assessment. In order to quantify the strength of dependence in the tails, one can use relevant quantities based on the limiting property of a copula through its diagonal. Let $C$ be a bivariate copula and $(U, V) \sim C$ be a bivariate uniform random vector with $C$ as the joint cumulative distribution function (cdf). Let $C$ be the survival function of $C$, and $\hat{C}$ be the associated survival copula. For a bivariate copula $C$, the lower tail dependence parameter is defined as $\lambda_L := \lim_{u \to 0^+} C(u, u)/u$, provided that the limit exists; similarly, the upper tail dependence parameter is defined as $\lambda_U := \lim_{u \to 0^+} \hat{C}(u, u)/u$, provided that the limit exists. More generally, Hua and Joe (2011b) uses tail order as a measure of strength of dependence in the joint upper or joint lower tails. For the bivariate upper tail, the tail order $\kappa$ is the exponent of $u$ in the tail expansion

$$\overline{C}(1 - u, 1 - u) \sim u^\kappa \ell(u), \quad u \to 0^+,$$

∗Corresponding author, hua@math.niu.edu, Division of Statistics, Northern Illinois University, DeKalb, IL, 60115, United States.
†Harry.Joe@ubc.ca, Department of Statistics, University of British Columbia, Vancouver, BC, V6T1Z4, Canada.
where $\kappa \geq 1$, and $\ell$ is a slowly varying function ($\ell$ could be a constant). For a measurable function $g : \mathbb{R}_+ \to \mathbb{R}_+$, if for any constant $r > 0$, $\lim_{x \to 0^+} g(rx)/g(x) = 1$, then $g$ is said to be slowly varying at 0+, denoted as $g \in \text{RV}_0(0^+)$; if for any constant $r > 0$, $\lim_{x \to \infty} g(rx)/g(x) = r^\alpha$; $\alpha \in \mathbb{R}$, then $g$ is said to be regularly varying at $\infty$ with variation exponent $\alpha$, and is denoted as $g \in \text{RV}_\alpha$. For a random variable $X$, when we say that $X$ is regularly varying at $\infty$, it actually means that the survival function $\overline{F} \in \text{RV}_\alpha$ with some variation exponent $\alpha < 0$. There is an analogy with $C(u, u)$ in the joint lower tail. With this concept, smaller $\kappa$ indicates more positive dependence in the tail: (a) $\kappa = 1$ for strongest dependence in the tail (usual tail dependence), (b) $1 < \kappa < 2$ which we call intermediate tail dependence, and (c) $\kappa = 2$ and slowly varying function $\ell$ with $\lim_{u \to 0^+} \ell(u) > 0$ which we call tail quadrant or tail orthant independence. It is possible for $\kappa > 2$ with negative dependence. The tail order is the reciprocal of the parameter $\eta$ defined in Ledford and Tawn (1996), and used in the extreme value literature.

Another way to investigate strength of dependence in the tails is through the limiting property along the boundaries of the copula. Let $C_{21}^1(\cdot|u) = \partial C(u, v)/\partial u$ be the conditional cdf of $[V|U = u]$ and let $C_{12}^1(\cdot|v) = \partial C(u, v)/\partial v$ be the conditional cdf of $[U|V = v]$. Strength of dependence in the upper tail can be studied through the cdfs of $[V|U = 1]$ and $[U|V = 1]$, that is, the conditional cdfs $C_{21}^1(\cdot|1)$ and $C_{12}^1(\cdot|1)$. If $C_{21}^1(\cdot|1)$ is a degenerate distribution at 1 or has positive probability at the point 1, then this is an indication of relatively stronger positive dependence in the upper tail. Likewise, the behavior of $C_{21}^1(\cdot|0)$ and $C_{12}^1(\cdot|0)$ gives an indication of strength of dependence in the lower tail. The functions $C_{12}^1(\cdot|0)$, $C_{12}^1(\cdot|1)$, $C_{21}^1(\cdot|0)$ and $C_{21}^1(\cdot|1)$ are called boundary conditional cdfs in the remainder of this paper.

Depending on the value of $\theta_U = \theta_{U,12} := \lim_{u \to 1^-} \{1 - C_{12}^1(u|1)\}$ and $\theta_L = \theta_{L,12} := \lim_{u \to 0^+} C_{12}^1(u|0)$, the strength of dependence in the upper [lower] tail can be (i) strongest if $\theta_U = 1$ and $\theta_L = 1$; (ii) intermediate if $0 < \theta_U < 1$ and $0 < \theta_L < 1$; (iii) weakest if $\theta_U = 0$ and $\theta_L = 0$. For the upper tail, these correspond to (i) $C_{12}^1(\cdot|1)$ has mass of 1 at 1; (ii) $C_{12}^1(\cdot|1)$ has positive but not unit mass at 1; (iii) $C_{12}^1(\cdot|1)$ has no mass at 1; and similar interpretation holds for the lower tail. By symmetry, we have the same classification for $C_{21}^1(\cdot|1)$ and $C_{21}^1(\cdot|0)$. However, for permutation asymmetric bivariate copulas, the tail behavior of $C_{12}^1$ and $C_{21}^1$ may be different. For a copula family such as the Gumbel family with upper tail dependence (upper tail order $\kappa = 1$), we notice that the conditional cdf $C_{12}^1(\cdot|1)$ is degenerate at 1. The condition of $C_{12}^1(\cdot|1)$ being degenerate at 1 also came up in Cooke et al. (2011) in the analysis of tail dependence of sums of random variables. Properties of $C_{12}^1(\cdot|1)$ were also considered in Hua and Joe (2012a) in the analysis of CTE of the form $E[X_1|X_2 = t]$ and $E[X_1|X_2 > t]$ as $t \to \infty$, where $X_1, X_2$ are dependent nonnegative random variables with copula $C$ and a common univariate distribution $F$.

Although the concepts of tail order and boundary cdfs can both capture the strength of dependence in the tails, they are based on limiting properties of copulas along different routes, and there are no general relationships between them due to great flexibility of copulas. However, we will try to make connections between the tail order and the boundary conditional cdfs for parametric copula families, and for copulas under certain conditions when possible.

Finally, we consider using CTEs to investigate the strength of dependence in the tails. CTEs for strength of dependence of a bivariate dependent random vector play a similar role as mean excess functions for univariate tail heavity. Mean excess functions of the form $E[X - t|X > t]$ can be used to distinguish the strength of univariate tail heavity (See Figure 6.2.4 of Embrechts et al. (1997)). Like the mean excess function, CTE of the form $E[X_1|X_2 > t]$ can be easily approximated empirically. So the study of the tail behavior of such
a CTE shall be helpful in developing relevant plots for visualizing the strength of dependence in the upper tails. Tail behavior of CTEs simply provides another way other than the usual tail dependence and tail order for looking at the strength of dependence in the tails. It is interesting to find connections between CTEs and the existing tail dependence concepts. However, the tail behavior of CTEs depends on both dependence structures and marginal distributions, the study here is much more complicated. For example, the usual tail dependence may lead to $\mathbb{E}[X_1 | X_2 > t] = O(t)$ as $t \to \infty$ for $F$ that has Pareto or power law tails with tail index $\alpha > 1$; see Zhu and Li (2012) and Hua and Joe (2011a) for relevant references. For a copula family such as the Frank family with tail quadrant independence (tail order $\kappa = 2$), we notice that $\mathbb{E}[X_1 | X_2 = t] = O(1)$ and $\mathbb{E}[X_1 | X_2 > t] = O(1)$ as $t \to \infty$ for different $F$ with finite mean, from direct calculations via Taylor expansions. We also have examples of intermediate tail dependence ($1 < \kappa < 2$) with $\mathbb{E}[X_1 | X_2 = t] = O(t^\gamma)$ and $0 < \gamma < 1$, where we use standard techniques for asymptotic approximations of integrals.

We study different tail behavior in detail only for bivariate copulas. Since some multivariate dependence structures can be built up from bivariate copulas, the study on bivariate cases may provide guidance for models. For example, understanding the properties on strength of dependence in the tails for bivariate copulas is useful for (a) choosing bivariate linking copulas in vines (b) choosing bivariate copulas for consecutive observations in a Markov time series model. Models based on the pair-copula construction or vine copula are applied in Aas et al. (2009) and Nikoloulopoulos et al. (2012); a reference for use of copulas for Markov time series models is Chapter 8 of Joe (1997). The paper is organized as the following. In Section 2, we establish possible combinations of tail order of form (a), (b), (c) and boundary conditional cdfs of form (i), (ii), (iii). The form of the boundary conditional cdfs also affects how we proceed to approximate CTEs. In Section 2.1, a general relation can be derived under the assumption of a positive dependence condition called stochastically increasing (SI) and additional structures in the copula. Sections 2.2 and 2.3 have results for the classes of bivariate extreme value and Archimedean copulas, respectively. Then in Section 3, we obtain conditions for $\mathbb{E}[X_1 | X_2 > t]$ and $\mathbb{E}[X_1 | X_2 = t]$ to be asymptotically $O(t), O(1)$ or $O(t^\gamma)$ where $0 < \gamma < 1$; we also refer to these three cases as asymptotically linear, constant and sub-linear, respectively. Finally, Section 4 concludes the paper with possible future research.

2 Boundary conditional cdfs of copulas

2.1 Overview

In this subsection, we discuss and prove some preliminary results with the positive dependence condition of SI.

The tail order for a bivariate copula involves only the upper corner near $(1, 1)$ or the lower corner near $(0, 0)$. However, if the horizontal line represents $X_2$ and the vertical line represents $X_1$, then the properties of $C_{2|1}(\cdot|1)$ depend on the copula near the top edge of the unit square, and similarly the properties of $C_{1|2}(\cdot|1)$, $C_{2|1}(\cdot|0)$, $C_{1|2}(\cdot|0)$ depend respectively on the copula near the right edge, bottom edge, and left edge of the unit square.

Because in general a copula can be pieced together from bivariate uniform distributions on different non-overlapping subsquares of the unit square, the form of $C_{1|2}(\cdot|1)$ can be quite varied as a mix of continuous and discrete components.

In order to get relations of the upper tail order $\kappa$ and $C_{1|2}(\cdot|1)$ (and analogous relations for the opposite...
corner), we assume a positive dependence condition such as SI. SI is a positive dependence concept for two random variables. \(X_1\) is SI in \(X_2\) if \(\Pr(X_1 > x | X_2 = t)\) is increasing in \(t\) for all \(x\). If \(C\) is the copula for \((X_1, X_2)\), then this is the same as \(C_{1|2}(u|v)\) decreasing in \(v\) for all \(0 < u < 1\). Similarly, \(X_2\) is SI in \(X_1\) if \(C_{2|1}(v|u)\) is decreasing in \(u\) for all \(0 < v < 1\). It is asymmetric in that \(X_1\) SI in \(X_2\) and \(X_2\) SI in \(X_1\) in general have to be established separately. If \(X_1, X_2\) are exchangeable (or have copula that is permutation symmetric), then one can just say that \(X_1, X_2\) satisfy the SI condition.

**Proposition 1** Suppose \(C\) is a bivariate copula that is SI for both conditionings. If \(C\) has lower tail dependence with \(\lambda_L > 0\), then the conditional cdfs \(C_{1|2}(\cdot|0)\) and \(C_{2|1}(\cdot|0)\) have strictly positive mass \(p_0\) at 0, and \(p_0 \geq \lambda_L\); if \(C\) has upper tail dependence with \(\lambda_U > 0\), then the conditional cdfs \(C_{1|2}(\cdot|1)\) and \(C_{2|1}(\cdot|1)\) have strictly positive mass \(p_1\) at 1, and \(p_1 \geq \lambda_U\).

**Proof:** Suppose \(U, V\) are uniformly distributed on \([0, 1]\), respectively, and let \(C\) be the copula of \((U, V)\). From the SI assumption, for any \(0 \leq w \leq 1\), \(\Pr(V \leq w | U = u)\) is decreasing in \(u\). Then

\[
\Pr(V \leq w | U = u) = w^{-1} \int_0^w \Pr(V \leq w | U = u) \, du \leq \Pr(V \leq w | U = 0).
\]

If \(C\) has lower tail dependence, then letting \(w \to 0\) on both sides of (1) leads to:

\[0 < \lambda_L \leq \Pr(V \leq 0 | U = 0).\]

So the conditional cdf \(C_{2|1}(\cdot|0)\) has strictly positive mass at 0. Similarly, we can prove the results for \(C_{1|2}(\cdot|0)\), \(C_{2|1}(\cdot|1)\) and \(C_{1|2}(\cdot|1)\).

For some commonly used parametric bivariate copula families with different tail patterns, we summarize results in Table 1 on tail order and the boundary conditional cdfs.

Bivariate extreme value and Archimedean copulas have a lot of structures through either the extreme value exponent function or the Laplace/Williamson transform functions, so that general results about the boundary conditional cdfs are possible and connections with the tail order can be made. In the following Sections 2.2 and 2.3, we will study the extreme value copula and the Archimedean copula, respectively. Heuristic arguments for elliptical copulas are in Section 2.4, and finally a summary Table 2 is given for the results we obtained in this section.

### 2.2 Bivariate extreme value copulas

For bivariate extreme value copulas, the property of SI is proved in [Guillen (2000)](Guillen2000). As extreme value copulas need not be permutation symmetric, the SI property means that one variable is SI in the other.

For a bivariate extreme value copula \(C\), we can write

\[
C(u, v) = C_A(u, v) = \exp\{-A(-\log u, -\log v)\},
\]

where \(A : [0, \infty)^2 \to [0, \infty)\) is convex, homogeneous of order 1 and satisfies \(\max(x_1, x_2) \leq A(x_1, x_2) \leq x_1 + x_2\) [Pickands (1981)]. Let \(B(w) := A(w, 1 - w)\) for \(0 \leq w \leq 1\), then \(B\) is convex and \(\max\{w, 1 - w\} \leq B(w) \leq 1\). Because \(B\) is convex, \(B\) is continuous and has 1-sided derivatives including at the boundaries of 0 and 1.
Table 1: Tail order and boundary conditional cdf

| Copula   | $\kappa_L$ | $C_{1|2}(u|0)$ | $C_{1|2}(u|1)$ |
|----------|------------|----------------|----------------|
| Gaussian | $2/(1 + \theta)$ | $1$ for $0 < u \leq 1$ | $T_{\nu+1}^{-1}(\phi \sqrt{\nu + 1} / \sqrt{1 - \phi^2})$ for $0 < u < 1$ |
| Student $t$ | $1$ | $(1 - e^{-\delta u})/(1 - e^{-\delta})$ for $0 \leq u \leq 1$ | $(1 - e^{-\delta u})e^{-\delta}/[(1 - e^{-\delta})e^{-\delta u}]$ for $0 \leq u \leq 1$ |
| Frank   | $2$ | $1$ for $0 < u \leq 1$ | $u^{1+\delta}$ for $0 \leq u \leq 1$ |
| Gumbel  | $2^{1/\delta}$ | $1$ for $0 < u \leq 1$ | $u^{1+\delta}e^{\delta(1-u^{-\delta})}$ for $0 \leq u \leq 1$ |
| MTCJ    | $1$ | $1$ for $0 < u \leq 1$ | $1$ for $0 < u \leq 1$ |
| BB2     | $2$ | $1$ for $0 < u \leq 1$ | $1$ for $0 < u \leq 1$ |

$T_{\nu}$ is the Student cdf with parameter $\nu > 0$.

2. The multivariate distribution and its copula has been derived independently by several authors and is attributed to several names, including Clayton. We refer to Joe et al. (2010) for justification for the term “MTCJ”.

what follows, we assume that $A$ and $B$ are differentiable to avoid technicalities, and on the boundaries the derivatives are the corresponding 1-sided derivatives. The conditions on $B$ imply that $B(0) = B(1) = 1$, $-1 \leq B'(0) \leq 0$ and $0 \leq B'(1) \leq 1$, where $B'$ is the left-hand derivative of $B$ for $B'(1)$ and the right-hand derivative of $B$ for $B'(0)$.

Proposition 2 (Upper and lower tails) Suppose $C$ is a bivariate extreme value copula as in [2], then

1. $C_{1|2}(u|1) = [1 - B'(1)]u$ for $0 \leq u < 1$;
2. $C_{1|2}(u|0) = u^{1+B'(0)}$ for $0 < u \leq 1$;
3. $C_{2|1}(v|1) = [1 + B'(0)]v$ for $0 \leq v < 1$;
4. $C_{2|1}(v|0) = v^{1-B'(1)}$ for $0 < v \leq 1$.

Proof: Write

$$C_{1|2}(u|v) = C(u, v)A_2(-\log u, -\log v)v^{-1}$$
$$= \exp(-A(-\log u, -\log v))A_2(-\log u, -\log v)v^{-1}$$
$$= \exp(\log v \cdot A(\log u/\log v, 1))A_2(1, \log v/\log u)v^{-1},$$

where $A_2$ is the partial derivative of $A(x_1, x_2)$ with respect to $x_2$.

(Upper tail) Since $A$ is homogeneous of order 0, $A_2$ is homogeneous of order 0. Then $C_{1|2}(u|1) = uA_2(-\log u, 0) = uA_2(1, 0)$. Since $B(w) := A(w, 1 - w)$, and thus letting $y := (1 - w)/w$, we have $A(1, y) = (1+y)B(1/(1+y))$. Therefore, $A_2(1, y) = B(1/(1+y))-(1+y)^{-1}B'(1/(1+y))$, and $A_2(1, 0) = B(1) - B'(1) = 1 - B'(1)$. So $C_{1|2}(u|1) = [1 - B'(1)]u$. 

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(Lower tail) Let $A_1$ be the partial derivative of $A(x_1, x_2)$ with respect to $x_1$. $A(z, 1) = (1 + z)B(z/(1 + z))$, and $A_1(z, 1) = B(z/(1 + z)) + (1 + z)^{-1}B'(z/(1 + z))$. So, $A_1(0, 1) = B(0) + B'(0) = 1 + B'(0)$. Note that $\lim_{u \to 0^+} A_2(1, \log v/\log u) = A_2(1, \infty) = B(0) = 1$. Therefore, letting $t = 1/\log(v)$, then as $v \to 0^+$, i.e., as $t \to 0^-$, by l'Hôpital’s rule,

$$
\log(C_{1|2}(u|v)) = \log(v \cdot A(\log u/\log v, 1) + \log(A_2(1, \log v/\log u)) - \log(v) = \frac{A(t \log u, 1) - 1}{t} + \log\left(A_2\left(1, \frac{1}{t \log u}\right)\right).
$$

Hence, $C_{1|2}(u|0) = u^{1+B'(0)}$.

Similarly, we can prove the results for the opposite conditioning. □

Remark 1 Proposition 2 implies that, $C_{1|2}(u|1)$ has mass $B'(1)$ at $u = 1$ if $B'(1) > 0$, and is degenerate at 1 if $B'(1) = 1$; $C_{1|2}(u|0)$ is degenerate at $u = 0$ with mass 1 if $B'(0) = -1$, and otherwise it has no mass at 0. Also, $C_{2|1}(cdot|1)$ is degenerate at 1 if $B'(0) = -1$ and $C_{2|1}(cdot|0)$ has a mass of 1 at 0 if $B'(1) = 1$.

Remark 2 The class of bivariate extreme value copulas is illuminating to show what is happening in the tails. For any bivariate extreme value copula, $\kappa_L = A(1, 1) = 2B(\frac{1}{2}) \in [1, 2]$ (Hua and Joe 2011b; if the bivariate extreme value copula is not the independence copula, then $\kappa_L = 1$ and $\kappa_U = 2 - A(1, 1) = 2 - 2B(\frac{1}{2})$. However, the boundary conditional cdfs $C_{2|1}(cdot|0)$, $C_{2|1}(cdot|1)$, $C_{1|2}(cdot|0)$, $C_{1|2}(cdot|1)$ depend on $B'(0)$ and $B'(1)$. That is, the strength of lower tail order and upper tail dependence depends on $B(\frac{1}{2})$, and the strength of dependence in the edges of the unit square depends on $B'(0)$ or $B'(1)$. The conditions of $B'(1) = 1$ and $B'(0) = -1$ hold for the Gumbel, Galambos and Hüsler-Reiss copula families (except the boundary case of independence). But the conditions do not hold for the $t$-EV bivariate extreme value copula (Demarta and McNeil 2005; Nikoloulopoulos et al. 2009), which is the extreme value limit of the bivariate $t$ copula.

### 2.3 Archimedean copulas

A bivariate Archimedean copula can be constructed as

$$
C(u, v) = C_\psi(u, v) := \psi(\psi^{-1}(u) + \psi^{-1}(v)),
$$

where $\psi(\infty) = 0$, $\psi(0) = 1$, and $\psi$ is 2-times monotone (nonincreasing and convex). $\psi$ can be a Williamson’s $d$-transform ($d \geq 2$) of a positive radial random variable $R$ (McNeil and Neslehová 2009) or a Laplace transform (LT) of a positive random variable. If $\psi$ is a LT such that $\psi(s) = 0$, then shows that $C_\psi$ satisfies the dependence concept of TP2 and thus SI.

**2.3.1 Usual tail dependence**

**Proposition 3** (Lower tail) Consider a bivariate Archimedean copula $C = C_\psi$ in (4). If $\psi \in RV_{-\alpha}$ with $\alpha \geq 0$, then the conditional distribution $C_{1|2}(:,0)$ is a cdf of a degenerate random variable at 0; that is, $C_{1|2}(u|0) = 1$ for any $0 < u \leq 1$. 

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Proof: Write

\[ C_{1|2}(u|0) = \lim_{v \to 0^+} \frac{\psi'(\psi^{-1}(u) + \psi^{-1}(v))}{\psi'(\psi^{-1}(v))} = \lim_{s \to \infty} \frac{\psi'(s + \psi^{-1}(u))}{\psi'(s)}. \quad (5) \]

Since \( \psi'(s) \) is increasing in \( s \), by the Monotone Density Theorem (Bingham et al. 1987, Theorem 1.7.2), \( \psi'(s) \sim (-\alpha)s^{-\alpha-1}\ell(s), \ s \to \infty \), for a slowly varying \( \ell(s) \). So \( -\psi'(0) \in \text{RV}_{-\alpha}, \) and thus for any \( 0 < u \leq 1 \),

\[ \lim_{s \to \infty} \frac{\psi'(s + \psi^{-1}(u))}{\psi'(s)} = 1, \text{ since for any small } \epsilon > 0, \text{ as } s \text{ is sufficiently large, } -\psi'((1 - \epsilon)s) \leq -\psi'(1 + \epsilon)s. \]

That is, \( C_{1|2}(u|0) = 1 \) for \( 0 < u < 1 \). \( \Box \)

Note that, the condition \( \psi \in \text{RV}_{-\alpha} \) with \( \alpha \geq 0 \) implies the usual lower tail dependence of \( C_\psi \). For upper tails, we have the following result.

**Proposition 4** *(Upper tail)* Consider a bivariate Archimedean copula \( C = C_\psi \) in \( \Psi \). If \( \psi'(0) = -\infty \), then the conditional distribution \( C_{1|2}(\cdot|1) \) is a cdf of a degenerate random variable at 1; that is, \( C_{1|2}(u|1) = 0 \) for any \( 0 \leq u < 1 \).

**Proof:** If \( \psi'(0) = -\infty \), then for any \( 0 \leq u < 1 \),

\[ C_{1|2}(u|1) = \lim_{v \to 1^-} \frac{\psi'(\psi^{-1}(u) + \psi^{-1}(v))}{\psi'(\psi^{-1}(v))} = \lim_{s \to 0^+} \frac{\psi'(s + \psi^{-1}(u))}{\psi'(s)} = 0, \]

since \( \psi'(\psi^{-1}(u)) < \infty \) for \( 0 \leq u < 1 \). \( \Box \)

Note that, the condition \( \psi'(0) = -\infty \) is a necessary condition for usual upper tail dependence of the Archimedean copula \( C_\psi \).

### 2.3.2 Intermediate tail dependence and tail quadrant independence

Another pattern of tail behavior other than the regularly varying tail is often needed when one studies the case of intermediate tail dependence. We refer to Bingham et al. (1987) for more details.

**Definition 1** A function \( g : \mathbb{R} \to (0, \infty) \) is \( \Gamma \)-varying if it is increasing and right-continuous, and there exists a measurable function \( h : \mathbb{R} \to (0, \infty) \) such that for any \( t \in \mathbb{R} \)

\[ \lim_{x \to \infty} \frac{g(x + th(x))}{g(x)} = e^t. \quad (6) \]

In this case, \( h(\cdot) \) is called an auxiliary function. If \( g \) is \( \Gamma \)-varying, then \( 1/g \) is said to be \( \Delta \)-varying, with the same auxiliary function \( h(\cdot) \).

A random variable \( X \) is said to belong to the maximum domain of attraction (MDA) of an extreme value distribution \( H \) if there exist normalizing constants \( \sigma_n > 0 \) and \( \mu_n \in \mathbb{R} \) such that

\[ (M_n - \mu_n)/\sigma_n \xrightarrow{d} H, \quad n \to \infty, \]

where \( M_n \) is the first order statistic (i.e., maximum) of a random sample of \( X \) with sample size \( n \), and \( \xrightarrow{d} \) means “convergence in distribution”. This is written as \( X \in \text{MDA}(H) \). We refer to Embrechts et al. (1997) for a classical reference on the concepts of MDA and general extreme value theory.
In the result below, MDA(Λ) or MDA(Gumbel) refers to the maximum domain of attraction of the Gumbel distribution \( \Lambda(x) = \exp\{-e^{-x}\} \) \((-\infty < x < \infty)\).

**Proposition 5** (Lower tail) Consider a bivariate Archimedean copula \( C = C_\psi \) in \([4]\). Let \( R \) be a nonnegative random variable with Williamson 2-transform \( \psi \) that satisfies \( \psi(0) = 1 \) (i.e., \( F_R(0) = 0 \)) and has auxiliary function \( h \). If \( R \in \text{MDA}(\Lambda) \) and \( h(s) \to \infty \) as \( s \to \infty \), then \( C_{1|2}(u|0) \) is a cdf of a random variable degenerate at 0; that is, \( C_{1|2}(u|0) \equiv 1 \) for any \( 0 < u \leq 1 \).

**Proof:** By Theorem 1 of Larsson and Nešlehová (2011), \( \psi \) is \( \Delta \)-varying. Since \( -\psi' \) is positive, decreasing and right-continuous, by Lemma 1 of Larsson and Nešlehová (2011), \( -\psi' \) is also \( \Delta \)-varying. For any \( 0 < u \leq 1 \), \( \psi^{-1}(u) < \infty \), and any small \( \epsilon > 0 \), since \( h(s) \to \infty \), as \( s \) is sufficiently large, \( ch(s) \geq \psi^{-1}(u) \). Therefore,

\[
e^{-\epsilon} = \lim_{s \to \infty} \frac{-\psi'(s + ch(s))}{-\psi'(s)} \leq C_{1|2}(u|0) := \lim_{s \to \infty} \frac{-\psi'(s + \psi^{-1}(u))}{-\psi'(s)} \leq -\frac{-\psi'(s)}{-\psi'(s)} = 1.
\]

Since \( \epsilon \) is arbitrarily small, we must have \( C_{1|2}(u|0) \equiv 1 \) for any \( 0 < u \leq 1 \). \( \square \)

**Remark 3** Let \( S \) denote the class of subexponential distributions (Embrechts et al., 1997), then \( S \cap \text{MDA}(\Lambda) \) satisfies the conditions in Proposition 5 (Example 3.3.35 of Embrechts et al., 1997). Note that, \( R \in \text{MDA}(\Lambda) \) together with an additional condition \( h \in \text{RV}_\alpha \) with \( \alpha \leq 1 \) just implies that the bivariate copula has lower tail order \( \kappa_L = 2^{1-\alpha} \) (see Proposition 7 of Larsson and Nešlehová, 2011 and Coles et al., 1999). If \( 0 < \alpha < 1 \), then the copula has lower intermediate tail dependence and \( h(s) \to \infty \), and thus \( C_{1|2}(u|0) \) is a degenerate cdf at 0. If \( \alpha = 0 \), it will lead to lower tail quadrant independence. For example, Erlang(\( d \))-simplex mixture leads to the independence copula (Remark 1 of McNeil and Nešlehová, 2010), and the corresponding auxiliary function for \( R \) is slowly varying (see Example 3.3.21 of Embrechts et al., 1997). Also, the auxiliary function \( h(x)/x \to 0 \) as \( x \to \infty \). So if \( \alpha = 1 \), then the corresponding slowly varying function for \( h \) shall converge to 0; it can be verified from Hua and Joe (2013) that this case corresponds to the boundary of usual tail dependence with the tail dependence parameter being 0.

**Remark 4** In Proposition 5 with the Williamson 2-transform, the auxiliary function for \( R \) can be chosen as the mean excess function of \( R \) (Embrechts et al., 1997); that is, \( h(x) = \mathbb{E}[R - x|R > x], x \in [0, \infty) \). So the condition \( h(s) \to \infty \) simply implies that the right tail of \( R \) is relatively heavier and thus leads to relatively stronger degree of dependence in the upper tail of \( R \times (S_1, S_2) \), where \((S_1, S_2)\) is uniformly distributed on the simplex \( \{s \geq 0 : s_1 + s_2 = 1\} \). Moreover, the survival copula is just an Archimedean copula (McNeil and Nešlehová, 2009), and therefore the degree of dependence in the lower tail of the Archimedean copula is relatively stronger, so that \( C_{1|2}(u|0) \) becomes degenerate at 0. When the right tail of \( R \) becomes even heavier, \( C_{1|2}(u|0) \) shall still degenerate at 0; that is simply the conclusion of Proposition 5.

Alternatively for a result with a condition that can be checked more easily, we make use of an assumption used in Hua and Joe (2011b):

\[
\psi(s) \sim T(s) = a_1 s^q \exp\{-a_2 s^r\} \quad \text{and} \quad \psi'(s) \sim T'(s), \quad s \to \infty, \quad \text{with} \quad a_1 > 0, a_2 \geq 0,
\]

where \( r = 0 \) implies \( a_2 = 0 \) and \( q < 0 \), and \( r > 0 \) implies \( r \leq 1 \) and \( q \) can be 0, negative or positive.

The assumption is adequate for applications to the Laplace transform families that are commonly used for Archimedean copulas, as well as other Laplace transform families that can be obtained by integration or...
Proposition 6 (Lower tail) Suppose a bivariate Archimedean copula $C$ satisfies (7). If $0 < r < 1$, then $C_{1/2}(0|0) = 1$. If $r = 1$, then $C_{1/2}(u|0) < 1$ for any $0 < u < 1$ and $C_{1/2}(0|0) = 0$.

Proof: For $0 < r \leq 1$, with assumption (7) and keeping the dominating term in

$$\psi'(s) \sim T'(s) \sim -a_1 a_2 s^{q+r-1} \exp\{-a_2 s^r\},$$

and for large $s$, the term on the right-hand side of (5) becomes

$$[s^{-1} \psi^{-1}(u) + 1]^{q+r-1} \exp\{-a_2 [\psi^{-1}(u) + s]^{r} + a_2 s^r\} \sim \exp\{-a_2 [\psi^{-1}(u) + s]^{r} + a_2 s^r\}$$

$$\sim \exp\{-a_2 s^{r} [1 + rs^{-1} \psi^{-1}(u) + o(s^{-2})] + a_2 s^r\} = \exp\{-a_2 [rs^{-1} \psi^{-1}(u) + o(s^{-2})]\}.$$ 

The limit as $s \to \infty$ is 1 if $0 < r < 1$. If $r = 1$, the limit is $\exp\{-a_2 \psi^{-1}(u)\} < 1$ and $\lim_{u \to 0^+} \exp\{-a_2 \psi^{-1}(u)\} = 0$. □

Unlike the lower tail, intermediate tail dependence of the upper tail of an Archimedean copula may lead to nondegenerate $C_{1/2}(\cdot|1)$.

Example 1 (copula in Joe-Ma (2000)) For bivariate Archimedean copulas constructed by normalized integral of the positive stable LT (see Example 4.2 in Joe and Ma (2000)),

$$\psi(s) = \int_{s}^{\infty} \exp\{-y^\eta\} dy/\Gamma(1 + 1/\eta) = G(s^\eta; \eta^{-1}), \ 0 < \eta < 1,$$

where $G$ is the survival function of Gamma$(1/\eta, 1)$ distribution with shape parameter $1/\eta$. Then from (8) below, $C_{1/2}(u|1) = \exp\{-G^{-1}(u; \eta^{-1})\}$, where $G^{-1}$ is the inverse function of $G$. So $C_{1/2}(\cdot|1)$ is a proper cdf.

For upper tails of Archimedean copulas, as we know in Proposition 4, $\psi'(0) = -\infty$ will lead to $C_{1/2}(\cdot|1)$ that is degenerate at 1, and thus every bivariate upper tail dependent Archimedean copula has such a property. Now, we want to know whether upper intermediate tail dependent Archimedean copulas can lead to degenerate $C_{1/2}(\cdot|1)$. From Proposition 6 of Hua and Joe (2013), if $R$ has Williamson 2-transform $\psi$ and $1/R \in MDA(\Phi_{\alpha})$ with $1 < \alpha < 2$ (that is, MDA of the Fréchet distribution $\Phi_{\alpha}(x) = \exp\{-x^{-\alpha}\}$, $x > 0$), then the bivariate Archimedean copula has upper intermediate tail dependence with tail order $\kappa = \alpha$. However, in this case, $-\infty < \psi'(0) < 0$ (see Hua and Joe (2013)) and then we can not get a degenerate $C_{1/2}(\cdot|1)$ by the proof of Proposition 4. As shown in Hua and Joe (2013), $\psi'(0) = -\infty$ may still occur for the boundary case $\alpha = 1$ if $\int_{0}^{1/s} \ell(1/y) y^{-1} dy \to \infty$ as $s \to 0^+$, where $\ell$ is the slowly varying function such that $F_R(x) := x\ell(x)$.

In terms of the upper tail order, we have the following result.

Proposition 7 (Upper tail) If the upper tail order is $\kappa_U$ and $1 < \kappa_U$, then $C_{1/2}(u|1) > 0$ for $0 < u < 1$. Furthermore $C_{1/2}(1|1) = 1$. If the expansion of $\psi$ at 0 is $\psi(s) \sim 1 - a_1 s + a_2 s^m$, $s \to 0$, where $m = \min\{2, \kappa_U\} \in (1, 2]$, then $1 - C_{1/2}(1 - y|1) = O(y^{m-1})$ as $y \to 0^+$. 

9
Proof: From results in Hua and Joe (2011b), if \( 1 < \kappa_U \) for a bivariate Archimedean copula, then \(-\infty < \psi'(0) < 0\). Then
\[
\lim_{s \to 0} \frac{\psi'(\psi^{-1}(u) + s)}{\psi'(s)} = \frac{\psi'(\psi^{-1}(u))}{\psi'(0)}.
\]
Therefore, \( 0 < C_{1|2}(u|v) < 1 \) for \( 0 < u < 1 \) and \( C_{1|2}(1^-|1) = 1 \).

From Hua and Joe (2011b), if \( \psi(s) \sim 1 - a_1s + a_2s^m \) where \( 1 < m < 2 \), then \( m = \kappa_U \) and if the tail order is greater than 2, then \( \psi(s) \sim 1 - a_1s + a_2s^2 \) is the expansion to order 2. Based on this expansion, \( \psi^{-1}(1 - \epsilon) = a_1^{-1}\epsilon + a_1^{-1-m}a_2\epsilon^m + o(\epsilon^m) \) as \( \epsilon \to 0 \) and
\[
\frac{\psi'(\psi^{-1}(1-y))}{\psi'(0)} = 1 - a_2a_1^{-m}my^{m-1} + o(y^{m-1}).
\]
Therefore \( 1 - C_{1|2}(1-y|1) = O(y^{m-1}) \).

The above result shows that for the case of \( C_{1|2}(1^-|1) = 1 \) (case (iii) of Table 2), the rate at which the conditional cdf approaches 1 does depend on the upper tail order.

### 2.4 Elliptical copulas

Any possible relationships between tail order and boundary cdfs for positive dependent bivariate elliptical copulas/distributions are not simple because the cdfs do not have closed forms in general. Here we just provide some heuristic arguments to show there are no simple relationship between the tail order and the form of boundary cdfs.

Let \( (X_1, X_2) \) have an elliptical distribution with dependence parameter \( 0 < \rho < 1 \) and let \( C \) be its copula. For properties of elliptical distributions, see Fang et al. (1990). We can write a stochastic representation for the conditional distributions:
\[
[X_1 | X_2 = y] \overset{d}{=} \rho y + \sqrt{1-\rho^2} \sigma(y) Z(y),
\]
where \( Z(y) \sim F_Z(\cdot; y) \) has a scale of 1 (on some measure of spread) and \( \sigma(y) \) is the (conditional) scale parameter. Then
\[
F_{1|2}(x|y) = P(X_1 \leq x | X_2 = y) = F_Z \left( \frac{x - \rho y}{\sigma(y)\sqrt{1-\rho^2}} ; y \right).
\]
If the scale parameter \( \sigma(y) \) is (asymptotically) constant or increasing in \( |y| \) at a sublinear rate, then \( C_{1|2}(\cdot|1) \) has mass of 1 at 1 and \( C_{1|2}(\cdot|0) \) has mass of 1 at 0. If the scale \( \sigma(y) \) is (asymptotically) increasing linearly in \( |y| \), then \( C_{1|2}(\cdot|0) \) and \( C_{1|2}(\cdot|1) \) are 2-point distributions with masses at 0 and 1. For the bivariate t distribution with parameter \( \nu > 0 \), \( Z(y) \) has the univariate t distribution with parameter \( \nu + 1 \) (for any \( y \)) and \( \sigma(y) = [(\nu + y^2)/(\nu + 1)]^{1/2} \).

For bivariate elliptical distributions, tail dependence occurs in all four corners if the survival function of the radial random variable has regularly varying tail (Embrechts et al. 2009; Schmidt 2002), and this corresponds to the boundary conditional cdfs of the corresponding copulas having masses on the set \{0, 1\}. We refer to Hua and Joe (2012b) for a discussion about the maximum possible tail dependence for a bivariate elliptical distribution. If the survival function of the radial random variable has rapidly varying tail, then the tail order \( \kappa_U = \kappa_L \) is in \((1, 2] \) for \( 0 < \rho < 1 \). In this case, the analysis for the type of boundary conditional cdfs is non-trivial.
2.5 Summary

To finish this section, a summary of the results is shown in Table 2. Note that, for Archimedean copulas that the boundary conditional cdfs behave quite differently for upper and lower intermediate dependence (tail order $1 < \kappa_U < 2$ or $1 < \kappa_L < 2$), and the bivariate t copula with positive dependence does not satisfy the SI condition.

Table 2: Possible relations between tail order and boundary conditional cdfs for Archimedean, extreme value, and normal/t copulas with positive dependence. Classification for the boundary conditional cdf: (i) boundary conditional cdf degenerate at the conditioning value; (ii) boundary conditional cdf with positive but not unit mass at the conditioning value; (iii) boundary conditional cdf with no mass at the conditioning value, where the boundary conditional cdf refers to one of $C_{1|2}(\cdot|0)$, $C_{1|2}(\cdot|1)$, $C_{2|1}(\cdot|0)$, $C_{2|1}(\cdot|1)$ depending on the context.

<table>
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<tr>
<td>(a) $\kappa = 1$</td>
<td>(ii)</td>
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3 Conditional tail expectation

Another way to capture the dependence in the tails is through the conditional tail expectations. Throughout this section, let continuous random variables $X_1, X_2$ be identically distributed with univariate cdf $F$ that is supported on $[0, \infty)$, $F_{12}$ be the joint cdf of $(X_1, X_2)$, and $C$ be the copula of $(X_1, X_2)$. Then $F_{12} = C(F_1, F_2) = \hat{C}(F_1, F_2)$, where $\hat{C}(u_1, u_2) = 1 - u_1 - u_2 + C(u_1, u_2)$ is a survival function, $\hat{C}(u_1, u_2) = u_1 + u_2 - 1 + C(1 - u_1, 1 - u_2)$ is a copula, referred to as the survival copula; also, $\mathbb{E}[X_j] = \int_0^\infty F(x) dx < \infty$ for $j = 1, 2$.

In some cases, we also assume that $C$ satisfies a positive dependence condition SI so that $\mathbb{E}[X_1|X_2 = t]$ and $\mathbb{E}[X_1|X_2 > t]$ are increasing in $t$; the former is simply due to the SI condition, and the latter is due to the fact that SI implies right-tail increasing (RTI) (Joe 1997, Theorem 2.3). We will also need properties of boundary conditional distributions discussed in Section 2 for deriving certain tail behavior of the CTEs. For example, in Section 3.3 a boundary conditional cdf with no mass at the corner point is needed in order to get a valid Taylor expansion.
We are interested in tail approximations to the CTEs
\[
\begin{align*}
E[X_1|X_2 > t] &= \int_0^\infty \frac{F_{12}(x,t)}{F(t)}
\frac{\partial}{\partial t} \left( \frac{\hat{C}(F(x),F(t))}{F(t)} \right) dx, \quad t \to \infty, \\
E[X_1|X_2 = t] &= \int_0^\infty \frac{\hat{C}_{12}(F(x)|F(t))}{F(t)} dx, \quad t \to \infty.
\end{align*}
\]
under (1) different tail conditions of the copula \( C \) such as the usual tail dependence, tail quadrant independence and intermediate tail dependence, (2) different tail conditions of \( \hat{C} \): MDA(\( \Phi_\alpha \)) and MDA(\( \Lambda \)). In the above, \( \hat{C}_{12}(u|v) = \partial \hat{C}(u,v)/\partial v \). Note that upper tail conditions on \( C \) correspond to lower tail conditions on \( \hat{C} \).

In Hua and Joe (2012a), \( E[X_1|X_2 > t] \) and \( E[X_1|X_2 = t] \) have been studied in terms of asymptotic comparisons under the condition of different tail dependence patterns. In this section, we study (checkable) conditions to determine whether the CTEs are \( O(t) \), \( O(t^\gamma) \) with \( 0 < \gamma < 1 \), or \( O(1) \) as \( t \to \infty \). With \( 0 < \gamma < 1 \), there could be a slowly varying function multiplying \( t^\gamma \). Note that, the rate of \( E[X_1|X_2 = t] \) and \( E[X_1|X_2 > t] \) as \( t \to \infty \) can not be faster than \( O(t) \) under very mild conditions of the univariate margins (see Section 3.1).

Figure 1: CTE plots for \( E[X_1|X_2 > t] \) with Pareto margins, and Gumbel, survival Gumbel and MTCJ copulas. The range of \( t \) in each plot was chosen to cover the support of \( t \) between the 1% and 99% quantiles.

We first use plots to illustrate comparisons between these three cases of usual tail dependence (\( \kappa = 1 \)), intermediate tail dependence (\( 1 < \kappa < 2 \)) and tail quadrant independence (\( \kappa = 2 \)) by Figure 1 for Pareto margins. The Blomqvist’s \( \beta \) for the three plots in Figure 1 were chosen to be \( \beta = 0.3, 0.6, 0.9 \), respectively. In each plot, the parameters for Gumbel, s.Gumbel and MTCJ copulas were calculated based on the same Blomqvist’s \( \beta \). For the Gumbel copula, \( \delta = 1.434, 2.484, 9.709 \), respectively for each case, and so does the \( \delta \) for the s.Gumbel copula. For the MTCJ copula, \( \delta = 0.863, 2.764, 13.513 \), respectively for each plot. The parameter for the identical Pareto margins were chosen to be \( \alpha = 100 \). If we use exponential univariate margins, then the patterns are very similar to those for Pareto margins with light tails.

Based on Figure 1 and similar plots with other Blomqvist \( \beta \) values and copula parameters \( \theta \), we find that when the upper tail order is 1 (Gumbel), the CTE plots seem to be linear in \( t \) on the whole support no matter what the \( \theta \) is. When the upper tail order is 2 (MTCJ), the CTE lines become very flat. The CTE plots for the intermediate upper tail dependence with the upper tail order \( 1 < \kappa < 2 \) (s.Gumbel) are located between
the above two cases. However, if the Pareto margins are too heavy-tailed with a very small parameter \( \theta \) (e.g., \( \theta = 1.1 \)), the CTE plots are not suitable for discriminating the three cases of tail orders. So, we choose a relatively large \( \theta \) for the purpose of discriminating the types of tail orders. Moreover, when Blomqvist’s \( \beta \) is very large (e.g., \( \beta = 0.9 \) in the third plot of Figure 1), it becomes very hard to distinguish tail dependence and intermediate tail dependence; similarly, when Blomqvist’s \( \beta \) is very small, the plot may becomes difficult to differentiate intermediate tail dependence and tail quadrant independence.

Similar patterns can be observed for CTE plots of \( \mathbb{E}[X_1|X_2 = t] \). However, unlike the former case, for \( \mathbb{E}[X_1|X_2 = t] \) it is relatively harder to get the empirical quantities. So CTE plots for \( \mathbb{E}[X_1|X_2 > t] \) may have a better potential to be further developed as a diagnostic tool for the patterns of dependence in the tails.

### 3.1 Upper bounds of asymptotic rate of CTEs

The main result in this subsection shows that with the condition that \( X_1 \) is SI in \( X_2 \) and a mild assumption on the distributions of \( X_1 \) and \( X_2 \), then the conditional expectation \( \mathbb{E}[X_1|X_2 = t] \) is increasing at a fastest rate of \( O(t) \).

#### Proposition 8

Suppose \( X_1, X_2 \) are identically distributed with the cdf being \( F \) that is supported on \([0, \infty)\) and the density function is \( f \), \( \mathbb{E}[X_1] < \infty \), and \( X_1, X_2 \) satisfy the SI condition of \( X_1 \) given \( X_2 \). Let \( C \) be the copula, and \( \hat{C} \) be the survival copula. If \( \lim \sup_{t \to \infty} \int_1^\infty F(x)dx/|tF(t)| < \infty \), then \( \mathbb{E}[X_1|X_2 = t] \) increases at a fastest rate of \( O(t) \) as \( t \to \infty \). With the \( \limsup \) condition but not necessarily the SI condition, the same conclusion holds for \( \mathbb{E}[X_1|X_2 > t] \).

**Proof:** Write

\[
\mathbb{E}[X_1|X_2 = t] = \int_0^\infty \hat{C}_{1|2}(F(x)|F(t))dx = \frac{1}{F(t)} \int_0^\infty \int_0^t \hat{C}_{1|2}(F(x)|F(t))dzdx
\]

\[
\leq \frac{1}{F(t)} \int_0^\infty \int_0^t \hat{C}_{1|2}(F(x)|z)dzdx = \frac{1}{F(t)} \int_0^\infty \hat{C}(F(x),F(t))dx = \mathbb{E}[X_1|X_2 > t]
\]

(11)

\[
\leq \frac{1}{F(t)} \int_0^\infty \min\{F(x),F(t)\}dx = \frac{1}{F(t)} \left[ \int_0^t F(t)dx + \int_t^\infty F(x)dx \right]
\]

\[
= t + \int_t^\infty \frac{F(x)}{F(t)}dx,
\]

where the first inequality follows from SI and the second inequality follows from the comonotonic copula as an upper bound. If \( \lim \sup \int_1^\infty F(x)dx/|tF(t)| < \infty \), then \( \mathbb{E}[X_1|X_2 = t] \) is bounded by \( O(t) \).

For \( \mathbb{E}[X_1|X_2 > t] \), the same conclusion holds with or without the SI condition, because the integral for \( \mathbb{E}[X_1|X_2 > t] \) appears in (11) above.

**Remark 5**

The condition \( \lim \sup \int_1^\infty F(x)dx/|tF(t)| < \infty \) in Proposition 8 is directly related to the von Mises condition on \( xF(x)/\hat{F}(x) \), and is satisfied by the distributions that belong to either MDA(\( \Lambda \)) or MDA(\( \Phi_\alpha \)). If \( X_1, X_2 \in \text{MDA}(\Lambda) \), then by Proposition 3.3.24 of Embrechts et al. (1997), \( \lim \int_1^\infty F(x)dx/|t\hat{F}(t)| = 0 \); if \( X_1, X_2 \in \text{MDA}(\Phi_\alpha) \), then by Corollary 3.3.8 and Proposition 3.3.9 of Embrechts et al. (1997), \( \lim \int_1^\infty F(x)dx/|t\hat{F}(t)| = 1/(\alpha - 1) < \infty \) when \( \mathbb{E}[X_1] < \infty \), i.e., \( \alpha > 1 \).
3.2 Limiting $O(t)$ conditional expectation

In this subsection, we will show that, under the condition of usual tail dependence, $E[X_1|X_2 > t]$ and $E[X_1|X_2 = t]$ are asymptotically proportional to $t$ as $t \to \infty$ (also known as, asymptotically linear in $t$).

Also, this pattern does not rely on the margins very much, and both MDA(Frédchet) and MDA(Gumbel) with finite expectations follow this pattern. The proportion is determined by the tail dependence function $b(w_1, w_2)$ introduced in [Joe et al. (2010)], or its partial derivatives. The tail dependence function is defined as $b(w_1, w_2) := \lim_{w_2 \to 0^+} C(uw_1, uw_2)/u$ for the lower tail, provided that the limit exists; $b(w_1, w_2) := \lim_{w_2 \to 0^+} \hat{C}(uw_1, uw_2)/u$ for the upper tail, provided that the limit exists.

The SI condition plays an important role in deriving the limit for $\hat{C}_{1|2}$. First, we give sufficient conditions that lead to the limit of $\hat{C}_{1|2}(uw_1|uw_2)$ as $u \to 0^+$ being the partial derivative of the associated tail dependence function.

**Lemma 9** Suppose $X_1, X_2$ are continuous random variables with copula $C$, and $X_1$ is SI in $X_2$. Let $\hat{C}$ be the survival copula, and $\hat{C}(uw_1, uw_2) \sim ub(w_1, w_2)$ as $u \to 0^+$, where $b$ is the lower tail dependence function of the copula $\hat{C}$ and $b(w_1, w_2)$ is partially differentiable with respect to $w_2$, then $\hat{C}_{1|2}(uw_1|uw_2) \sim b_{1|2}(w_1|w_2)$ as $u \to 0^+$, where $b_{1|2}(w_1|w_2) = \partial b(w_1, w_2)/\partial w_2$.

**Proof:** For a small $\epsilon > 0$, write

$$
\int_{w_2-\epsilon}^{w_2} \hat{C}_{1|2}(uw_1|ut)dt = \int_{u(w_2-\epsilon)}^{uw_2} \frac{\hat{C}_{1|2}(uw_1|y)}{u}dy = \frac{\hat{C}(uw_1, uw_2) - \hat{C}(uw_1, u(w_2-\epsilon))}{u}.
$$

Then the condition that $X_1$ is SI in $X_2$ implies that $\hat{C}(u|v)$ is nonincreasing in $v$, and then (12) implies that

$$
\limsup_{u \to 0^+} \hat{C}_{1|2}(uw_1|uw_2) \leq \liminf_{u \to 0^+} \frac{\hat{C}(uw_1, uw_2) - \hat{C}(uw_1, u(w_2-\epsilon))}{ue} = \frac{b(w_1, w_2) - b(w_1, w_2 - \epsilon)}{\epsilon}.
$$

Then letting $\epsilon \to 0^+$ leads to

$$
\limsup_{u \to 0^+} \hat{C}_{1|2}(uw_1|uw_2) \leq b_{1|2}(w_1|w_2).
$$

Similarly, considering the integration in (12) from $w_2$ to $w_2 + \epsilon$ will imply that $\liminf_{u \to 0^+} \hat{C}_{1|2}(uw_1|uw_2) \geq b_{1|2}(w_1|w_2)$. Therefore, $\hat{C}_{1|2}(uw_1|uw_2) \sim b_{1|2}(w_1|w_2)$ as $u \to 0^+$. \hfill \Box

If the univariate margins belongs to MDA(Frédchet) with finite expectation, then the following result holds for the CTEs being asymptotically proportional to $t$, when $t$ is sufficiently large.

**Proposition 10** (MDA of Frédchet) Suppose $X_1, X_2$ are identically distributed and $X_1, X_2 \in \text{MDA}(\Phi_\alpha)$ with $\alpha > 1$, and $C, \hat{C}$ are the copula and the survival copula of $(X_1, X_2)$, respectively. If $\hat{C}(uw_1, uw_2) \sim ub(w_1, w_2)$ as $u \to 0^+$, where $b$ is the lower tail dependence function of the copula $\hat{C}$; that is, $\lim_{u \to 0^+} \hat{C}(uw_1, uw_2)/u = b(w_1, w_2)$ pointwise for $w_1 > 0$, $w_2 > 0$. Then

$$
E[X_1|X_2 > t] \sim t \int_0^{\infty} b(w^{-\alpha}, 1) dw , \quad t \to \infty.
$$
If $X_1$ is SI in $X_2$ and $b_{1/2}(w_1|w_2) = \partial b(w_1, w_2)/\partial w_2$, then

$$E[X_1|X_2 = t] \sim t \int_0^\infty b_{1/2}(w^-|1) \, dw, \quad t \to \infty.$$  

Proof: Letting $w := x/t$, with a small $\epsilon > 0$ and a large $0 < s < \infty$, then

$$\frac{E[X_1|X_2 > t]}{t} = \int_0^\infty \frac{\hat{C}(F(wt), F(t)) \, dw}{F(t)} = \left( \int_{(0,\epsilon)} + \int_{[\epsilon, s]} + \int_{(s,\infty)} \right) \hat{C}\left(\frac{F(t)F(wt)}{F(t)}, F(t)\right) \, dw / F(t).$$

Since

$$\hat{C}\left(\frac{F(t)F(wt)}{F(t)}, F(t)\right) / F(t) \leq \min\{F(wt)/F(t), 1\},$$

and $F \in \text{RV}_{-\alpha}$ and thus $F(wt)/F(t) \to w^{-\alpha}$ uniformly in $w \in [1, \infty)$ as $t \to \infty$ (Bingham et al. (1987)),

$$\lim_{t \to \infty} \int_0^\infty \min\{F(wt)/F(t), 1\} \, dw = \lim_{t \to \infty} \left[ \int_0^1 1 \, dw + \int_1^\infty \{F(wt)/F(t)\} \, dw \right] = 1 + \int_1^\infty w^{-\alpha} \, dw = \frac{\alpha}{\alpha - 1} < \infty. \quad (13)$$

Then by the dominated convergence theorem,

$$\lim_{t \to \infty} \frac{E[X_1|X_2 > t]}{t} = \int_0^\infty \lim_{t \to \infty} \hat{C}\left(\frac{F(t)F(wt)}{F(t)}, F(t)\right) / F(t) \, dw = \left( \int_{(0,\epsilon)} + \int_{[\epsilon, s]} + \int_{(s,\infty)} \right) \lim_{t \to \infty} \hat{C}\left(\frac{F(t)F(wt)}{F(t)}, F(t)\right) / F(t) \, dw =: \eta_1(\epsilon) + \eta_2(\epsilon, s) + \eta_3(s).$$

Moreover, as $\epsilon < 1$ is sufficiently small

$$0 \leq \eta_1(\epsilon) \leq \int_{(0,\epsilon)} \lim_{t \to \infty} \min\{F(wt)/F(t), 1\} \, dw = \epsilon; \quad (14)$$

as $s > 1$ is sufficiently large, due to the uniform convergence for $F(wt)/F(t)$ in $w \in (s, \infty)$,

$$0 \leq \eta_3(s) \leq \int_{(s,\infty)} \lim_{t \to \infty} \min\{F(wt)/F(t), 1\} \, dw = \int_{(s,\infty)} w^{-\alpha} \, dw = \frac{s^{-\alpha + 1}}{\alpha - 1}. \quad (15)$$

Now consider $\eta_2$. Due to Theorem 1 of Schmidt and Stadtmüller (2006), that is, the convergence $\hat{C}(uw_1, uw_2)/u \to b(w_1, w_2)$ is locally uniform in $(w_1, w_2) \in \mathbb{R}^2_+$ (i.e., uniform convergence in any compact set in $\mathbb{R}^2_+$), we have

$$\eta_2(\epsilon, s) = \int_{[\epsilon, s]} b(w^{-\alpha}, 1) \, dw. \quad (16)$$
Letting $\epsilon \to 0$ and $s \to \infty$ in [14], [15] and [16] leads to
\[
\lim_{t \to \infty} \frac{\mathbb{E}[X_1 | X_2 > t]}{t} = \int_0^\infty b(w^{-1}, 1) \, dw.
\]
Note that, essentially the first part has also been proved in Proposition 1 of [Cai et al. 2012].

Similarly,
\[
\frac{\mathbb{E}[X_1 | X_2 = t]}{t} = t^{-1} \int_0^\infty \hat{C}_{1|2}(F(x) | F(t)) \, dx = \int_0^\infty \hat{C}_{1|2}(F(wt) | F(t)) \, dw
\]
\[
= \int_0^\infty \hat{C}_{1|2} \left( \frac{F(wt)}{F(t)} \right) \frac{F(t)}{F(t)} \, dw.
\]

Then, $X_1$ SI in $X_2$ implies from [11] that
\[
\frac{\hat{C}_{1|2}(F(x) | F(t))}{t} \leq \min\{F(x), F(t)\}.
\]

Then due to [13], the dominated convergence theorem can be applied here, and thus
\[
\lim_{t \to \infty} \frac{\mathbb{E}[X_1 | X_2 = t]}{t} = \int_0^\infty \lim_{t \to \infty} \hat{C}_{1|2} \left( \frac{F(wt)}{F(t)} \right) \frac{F(t)}{F(t)} \, dw.
\]

Similarly to the above arguments, now it remains to prove $\hat{C}_{1|2}(uw | u) \to b_{1|2}(w | 1)$ locally uniformly in $w \in \mathbb{R}_+$. Due to Theorem 1 of [Schmidt and Stadtmüller 2006], $b(w_1, w_2)$ is nondecreasing and Lipschitz continuous and thus $b_{1|2}(w_1 | w_2) \geq 0$ shall be bounded from above. For any compact set $K \subset \mathbb{R}_+^2$, define $M_u := \sup_{(w_1, w_2) \in K} |\hat{C}_{1|2}(uw_1 | uw_2) - b_{1|2}(w_1 | w_2)|$, and we shall have $M_u \to 0$ as $u \to 0^+$. Therefore, $\hat{C}_{1|2}(uw_1 | uw_2) \to b_{1|2}(w_1 | w_2)$ locally uniformly, which completes the proof. \qed

Next we study the case where the identical univariate margins belong to MDA($\Lambda$). A random variable $X$ with cdf $F$ in MDA($\Lambda$) if and only if there exists a positive auxiliary function $a(\cdot)$ such that
\[
\lim_{t \to \infty} \frac{\mathbb{E}[F(t + ya(t))]}{F(t)} = e^{-y}, \quad y \in \mathbb{R},
\]
where $a(\cdot)$ can be chosen as $a(t) = \int_t^\infty \{F(x) / F(t)\} \, dx$. We refer to [Embrechts et al. 1997] for a reference.

**Proposition 11 (MDA of Gumbel)** Suppose $X_1, X_2$ are identically distributed and $X_1, X_2 \in$ MDA($\Lambda$) with auxiliary function $a(\cdot)$, and $\hat{C}$ is the survival copula of $(X_1, X_2)$. Suppose $\hat{C}(uw_1, uw_2) \sim ub(w_1, w_2)$ as $u \to 0^+$, where $b$ is the lower tail dependence function of the copula $\hat{C}$, and $\hat{C}_{1|2}(uw_1 | uw_2) \sim b_{1|2}(w_1 | w_2)$ as $u \to 0^+$. Then
\[
\mathbb{E}[X_1 | X_2 > t] \sim b(\infty, 1) t, \quad t \to \infty.
\]
\[
\mathbb{E}[X_1 | X_2 = t] \sim b_{1|2}(\infty, 1) t, \quad t \to \infty,
\]
where the latter result assumes that $X_1$ is SI in $X_2$.  

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Proof: Letting \( y := (x - t)/a(t) \), as \( t \to \infty \),
\[
\frac{\mathbb{E}[X_1 | X_2 > t]}{t} = \frac{(a(t)/t) \int_{-t/a(t)}^{\infty} \hat{C}(F(t + ya(t)), F(t)) dy}{F(t)}
\]
\[
= \frac{(a(t)/t) \int_{-t/a(t)}^{\infty} \hat{C} \left( \frac{F(t + ya(t))}{F(t)}, F(t) \right) dy}{F(t)}.
\]
Since
\[
\hat{C} \left( \frac{F(t + ya(t))}{F(t)}, F(t) \right) / F(t) \leq \min \left\{ \frac{F(t + ya(t))}{F(t)}, 1 \right\},
\]
and \( 0 \leq a(t)/t \to 0 \) (Embrechts et al. (1997)) and thus for any small \( \epsilon > 0 \) when \( t \) is sufficiently large, \( 0 \leq a(t)/t \leq \epsilon \),
\[
\lim_{t \to \infty} \int_{-t/a(t)}^{\infty} \frac{a(t)}{t} \min \left\{ \frac{F(t + ya(t))}{F(t)}, 1 \right\} dy = \lim_{t \to \infty} \left[ \int_{-t/a(t)}^{0} \frac{a(t)}{t} dy + \int_{0}^{\infty} \frac{a(t)}{t} \frac{F(t + ya(t))}{F(t)} dy \right]
\]
\[
= 1 + \lim_{t \to \infty} \int_{0}^{\infty} \frac{a(t)}{t} \frac{F(t + ya(t))}{F(t)} dy \leq 1 + \epsilon \lim_{t \to \infty} \int_{0}^{\infty} \frac{F(t + ya(t))}{F(t)} dy
\]
\[
= 1 + \epsilon \int_{0}^{\infty} e^{-y} dy = 1 + \epsilon < \infty,
\] (18)
where the first equality in (18) is due to the fact that \( \frac{F(t + ya(t))}{F(t)} \to e^{-y} \) uniformly in \( y \in [-\infty, \infty] \) as \( t \to \infty \) (Proposition 3.10.2 of Bingham et al. (1987)). Therefore, letting \( w := ya(t)/t \) and applying the dominated convergence theorem leads to
\[
\lim_{t \to \infty} \mathbb{E}[X_1 | X_2 > t] = \lim_{t \to \infty} \left[ (a(t)/t) \int_{-t/a(t)}^{\infty} \hat{C} \left( \frac{F(t + ya(t))}{F(t)}, F(t) \right) dy \right] / F(t)
\]
\[
= \lim_{t \to \infty} \int_{-1}^{\infty} \hat{C} \left( \frac{F(t + wt)}{F(t)}, F(t) \right) / F(t) dw
\]
\[
= \int_{-1}^{\infty} b(1 + w)^{-1} dw = \int_{-1}^{0} b(\infty, 1) dw = b(\infty, 1),
\] (19)
where the first equality in (19) is also due to Theorem 1 of Schmidt and Stadtmüller (2006) and the fact that \( F \in RV_{-\infty} \) the class of rapidly varying functions.

For \( \mathbb{E}[X_1 | X_2 = t] \) assuming \( X_1 \) is SI in \( X_2 \), then from (11), the above implies that the dominated convergence theorem can be used to conclude:
\[
\lim_{t \to \infty} \mathbb{E}[X_1 | X_2 = t] = \lim_{t \to \infty} \int_{-1}^{\infty} \hat{C}_{1|2}(F(t + wt), F(t)) dw = \lim_{t \to \infty} \int_{-1}^{\infty} \hat{C}_{1|2} \left( \frac{F(t + wt)}{F(t)}, F(t) \right) dw
\]
\[
= \int_{-1}^{\infty} b_{1|2}(1 + w)^{-1} dw = b_{1|2}(\infty, 1).
\]
\[
\square
\]

Remark 6 We remark on bounds for \( b(\infty, 1) \) and \( b_{1|2}(\infty, 1) \) in the above result. By definition, \( b(w, 1) = \lim_{u \to 0} C(uw, u)/u \leq \lim_{u \to 0} u/u = 1 \) for \( w > 1 \), and \( b(w, 1) \) is increasing in \( w \). Hence \( 0 < b(\infty, 1) \leq 1 \).
if tail dependence exists. For an exponent that mixes those of the Galambos and independence copulas, 
\[ A(w_1, w_2) = (1 - \eta)[w_1 + w_2 - (w_1^{-\theta} + w_2^{-\theta})^{-1/\theta}] + \eta(w_1 + w_2), \]
where \( 0 < \eta < 1 \) and \( \theta > 0 \), then 
\[ b(w_1, w_2) = (1 - \eta)(w_1^{-\theta} + w_2^{-\theta})^{-1/\theta} \]
and \( b(\infty, 1) = 1 - \eta < 1 \). Also \( b_{1|2}(w|1) = \lim_{u \to 0} \hat{C}_{1|2}(uw|u) \) in increasing in \( w \) and \( 0 < b_{1|2}(\infty|1) \leq 1 \) if tail dependence exists.

### 3.3 Limiting \( O(1) \) conditional expectation

From Figure 1, we can find that the CTE line for the tail quadrant independence case (MTCJ) tends to converge to a finite number, if the parameter \( \theta \) for the Pareto margins is sufficiently large.

From [9] and [10], \( E[X_1|X_2 > t] = \int_0^\infty \frac{\hat{C}(F(x), F(t))}{F(t)} \, dx \) and \( E[X_1|X_2 = t] = \int_0^\infty \hat{C}_{1|2}(F(x)|F(t)) \, dx \). If \( \hat{C}_{1|2}(|0) \) is a continuous distribution on (0,1) with no mass at 0 (case (iii) of Table 2 in Section 2.1), then for any given \( 0 < u < 1 \) one can take Taylor expansions of \( \hat{C}(u, v) \) and \( \hat{C}_{1|2}(u|v) \) about \( v = 0 \) to show that the two CTEs are \( O(1) \) provide that Taylor remainder term satisfies certain conditions. However, the expansions are invalid for cases (i) and (ii) of Table 2.

**Proposition 12** Suppose \( X_1 \) and \( X_2 \) are nonnegative continuous random variables with copula \( C \) and \( E[X_1] < \infty \). Let \( \hat{C} \) be the survival copula. If

1. there exists \( 0 < v_0 < 1 \) and \( 0 < k < \infty \) such that \( u, v \leq v_0 \) implies that \( \hat{C}(u, v) \leq kvu \), and,
2. \( \hat{C}_{1|2}(F(.)|0) \) is the survival function of a random variable \( Y \) with \( E[Y] < \infty \),

then \( E[X_1|X_2 > t] \to E[Y] \). In addition, if \( X_1 \) is SI in \( X_2 \), then \( E[X_1|X_2 = t] \to E[Y] \).

**Proof:** From condition 1, there exists a \( t_1 < \infty \) such that \( t \geq t_1 \) implies that

\[
\int_0^\infty \frac{\hat{C}(F(x), F(t))}{F(t)} \, dx \leq \int_0^{v_0} 1 \, dx + \int_{v_0}^{\infty} kF(x) \, dx < \infty,
\]

where the first inequality is due to \( \hat{C}(F(x), F(t)) \leq \min\{F(x), F(t)\} \), and the second inequality is due to \( k < \infty \) and \( E[X_1] < \infty \). Therefore, the dominated convergence theorem can be applied to get

\[
\lim_{t \to \infty} \int_0^\infty \frac{\hat{C}(F(x), F(t))}{F(t)} \, dx = \int_0^\infty \lim_{t \to \infty} \frac{\hat{C}(F(x), F(t))}{F(t)} \, dx.
\]

For any given \( 0 < u < 1 \), the Taylor expansion of \( \hat{C}(u, v) \) about \( v = 0 \) is

\[
\hat{C}(u, v) = \hat{C}(u, 0) + v\hat{C}_{1|2}(u|0) + R(u, v) = v\hat{C}_{1|2}(u|0) + o(v),
\]

where \( R(u, v) = o(v) \) is the remainder. Then, by (20),

\[
\lim_{t \to \infty} \int_0^\infty \frac{\hat{C}(F(x), F(t))}{F(t)} \, dx = \int_0^\infty \hat{C}_{1|2}(F(x)|0) \, dx = E[Y],
\]

since \( \hat{C}_{1|2}(F(x)|0) \) is the survival function of \( Y \) that is supported on \([0, \infty)\).
With a similar argument in the proof of Proposition 8, the SI condition implies that the dominated convergence theorem is valid and
\[
\lim_{t \to \infty} \int_0^\infty \tilde{C}_{1/2}(\tilde{F}(x)\tilde{F}(t)) \, dx = \int_0^\infty \lim_{t \to \infty} \tilde{C}_{1/2}(\tilde{F}(x)\tilde{F}(t)) \, dx = \int_0^\infty \tilde{C}_{1/2}(\tilde{F}(x)) \, dx = E[Y].
\]
Hence, \(E[X_1|X_2 = t] \to E[Y]\), for the \(Y\) given above.

**Remark 7** The conditions on the copula in Proposition 12 are easy to verify and can be shown to apply for common bivariate copula families \(C\) with upper tail order 2 (i.e., \(\tilde{C}\) with lower tail order 2). From Table 1, upper tail quadrant independent copula families such as Frank and MTCJ all satisfy the conditions. Note that the condition 1 in Proposition 12 rules out the cases where the lower tail order of \(\tilde{C}\) is \(\kappa < 2\), or \(\kappa = 2\) and the associated slowly varying function \(\ell(u) \to \infty\) as \(u \to 0^+\).

In general, this result can apply for copula families in the combination of (c) and (iii) of Table 2 if the associated slowly varying function converges to a finite value, but not for the combination of (b) and (iii), although for which the boundary conditional cdf may also be proper. The techniques used to derive the tail behavior of the CTEs under the condition of \(1 < \kappa < 2\) will be in the next subsection.

### 3.4 Intermediate \(O(t^\gamma)\) conditional expectation

When the upper tail order of \(C\) (lower tail order of \(\tilde{C}\)) is \(1 < \kappa < 2\), the tail behavior of the CTEs becomes more subtle, and it depends on the tail behavior of the univariate margins and the copula as well. In this subsection, we first present procedures that are based on asymptotic approximation techniques for the tail behavior of \(\tilde{C}\) and \(C\), where the tail pattern of CTEs is \(O(t^\gamma)\) for \(0 < \gamma < 1\). Then concrete examples with intermediate upper tail dependence (survival Gumbel copula) and power-law or (sub) exponential-law margins will be given in Examples 5 and 6 to illustrate the subtle behavior of the CTEs.

Essentially, there are methods used in asymptotic approximations of integrals (see Wong (1989) and Breitwisch (1994)) that can apply after appropriate transforms of the integrating variable.

Suppose that \(X_1, X_2\) have the identical marginal distribution with cdf \(F\) being absolutely continuous and density \(f = F'\). Also, \(X_1\) has finite mean (i.e., \(\int_0^\infty F(x) \, dx < \infty\)). Now, we present the procedures that are used to derive the pattern of tail behavior of the CTEs in what follows:

- **Transform** \(F(t) \to e^{-T}\) (or \(T = -\log F(t)\)) and \(F(x) \to e^{-y}\) (or \(y = -\log F(x)\)) to get:
  \[
  E[X_1|X_2 > t] = \int_0^\infty e^T \tilde{C}(e^{-y}, e^{-T})e^{-y}[f(F^{-1}(1 - e^{-y}))]^{-1}\, dy.
  \]

- **Transform** \(y = sT\) to get
  \[
  E[X_1|X_2 > t] = T \int_0^\infty e^{sT} \tilde{C}(e^{-sT}, e^{-T})e^{-sT}[f(F^{-1}(1 - e^{-sT}))]^{-1}\, ds. \tag{21}
  \]

- Similarly, with the two steps,
  \[
  E[X_1|X_2 = t] = T \int_0^\infty \tilde{C}_{1/2}(e^{-sT}|e^{-T})e^{-sT}[f(F^{-1}(1 - e^{-sT}))]^{-1}\, ds. \tag{22}
  \]
• The integrand of (21) or (22) can be written as \( \exp\{Tg(s; T) + \log h(s)\} = \exp\{Tg(s; T)\}h(s) \), where \( h(s) > 0 \) and \( Tg(s; T) \) is the dominant term of the exponent. The function \( g \) depends on \( \log \hat{C} \) or \( \log \hat{C}_{1|2} \) and \( f \in F^- \).

After the change of variables in the integral, the following procedure can be attempted for any margin \( \hat{F} \) and any bivariate copula \( \hat{C} \), to get conditions so that approximations to the above integrals (21) and (22) apply.

• Check if \( g(0; T) = 0 \) and \( g(\infty; T) = -\infty \), and if so, proceed to one of the next two possibilities.

• If \( g'(0; T) > 0 \) as \( T \to \infty \) and Laplace’s method (the Laplace approximation) is applicable, the asymptotic rate might be 
\[
T^{1/2}e^{T\gamma}h(s_0(T))[\log g''(s_0(T); T)]^{-1/2},
\] where \( \gamma = \lim_{T \to \infty} \max_s g(s; T) \), \( s_0(T) = \arg \max_s g(s; T) \), \( g'' \) exists and is continuous at this mode, and \( -g''(s_0(T); T) \) has a positive limit as \( T \to \infty \). This comes from
\[
T \int_0^\infty e^{Tg(s; T)}h(s) \, ds \sim T \int_0^\infty h(s) \exp\{Tg(s_0(T); T) + \frac{1}{2}(s - s_0(T))^2g''(s_0(T); T)\} \, ds
\]
\[
\sim T e^{Tg(s_0(T); T)}h(s_0(T))\sqrt{\frac{2\pi}{-Tg''(s_0(T); T)}}.
\]

Note that the Laplace approximation just involves renormalization with a \( N(\mu, \sigma^2) \) density where \( \sigma \to 0 \).

• If \( g(s; T) \) is decreasing in \( s \), transform \( z = -g(s; T), s = m(z; T) \) and

\[
T \int_0^\infty e^{Tg(s; T)}h(s) \, ds = T \int_0^\infty e^{-Tz}h(m(z; T)) m'(z; T) \, dz.
\]

Then Watson’s lemma (e.g., Wang (1989)) might be applicable, and the asymptotic rate depends on the analytic form of \( h(m(z; T)) \) \( m'(z; T) \) near \( z = 0 \).

The above procedure may also work in some cases for Section 3.3 to derive the \( O(1) \) tail pattern of the CTEs as \( t \to \infty \). However, with the usual tail dependence for Section 3.2, the Laplace approximation might fail because \( g \) is not differentiable at the mode in the limit.

Examples 2 and 3 below show what the transforms are like for Pareto and exponential distributions. Also some details of \( \hat{C}(e^{-sT}, e^{-T}) \) and \( \hat{C}_{1|2}(e^{-sT}|e^{-T}) \) are given in Example 4 for the Gumbel and extreme value copulas.

Example 2 (Pareto margin) For Pareto(\( \alpha \)) with \( \alpha > 1 \), \( y = \alpha \log(1 + x), x = e^{y/\alpha} - 1, dx = \alpha^{-1}e^{y/\alpha}dy, T = \alpha \log(1+t) \).

(i) The integral for \( \mathbb{E}[X_1|X_2 \geq t] \) is \( \alpha^{-1} \int_0^\infty e^{T\hat{C}(e^{-y}, e^{-T})}e^{y/\alpha}dy = \alpha^{-1} T \int_0^\infty e^{T\hat{C}(e^{-sT}, e^{-T})}e^{sT/\alpha}ds \), and \( g(s; T) = 1 + s/\alpha + T^{-1} \log \hat{C}(e^{-sT}, e^{-T}) \).

(ii) The integral for \( \mathbb{E}[X_1|X_2 = t] \) is \( \alpha^{-1} T \int_0^\infty \hat{C}_{1|2}(e^{-sT}|e^{-T})e^{sT/\alpha}ds \), and \( g(s; T) = s/\alpha + T^{-1} \log \hat{C}_{1|2}(e^{-sT}|e^{-T}) \).

For \( \mathbb{E}[X_1|X_2 \geq t] \), note that \( g(0; T) = 0 \) and \( g(\infty; T) \leq 1 + \lim_{s \to \infty}[s/\alpha - s] \). The latter is \( -\infty \) if \( \alpha > 1 \) (the condition for the Pareto distribution to have finite mean). Furthermore,

\[
g'(s; T) = \alpha^{-1} - e^{-sT}\hat{C}_{2|1}(e^{-T}|e^{-sT})/\hat{C}(e^{-sT}, e^{-T}),
\]
and \( g'(0; T) \) behaves like \( \alpha^{-1} - e^{-T}\hat{C}_{2|1}(e^{-T}|1) \) as \( T \to \infty \) or \( \alpha^{-1} - \nu^{-1}\hat{C}_{2|1}(\nu|1) \) as \( \nu \to 0 \). The sign of \( g'(0; T) \) depends on \( \alpha \) and the copula \( \hat{C} \).

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For $\mathbb{E}[X_1 | X_2 = t]$, the limits $g(\infty; T)$ and $g'(0; T)$ require analysis in specific cases of $\hat{C}_{1|2}(e^{-sT}|e^{-T})$.

**Example 3 (Weibull margin)** Let $\bar{F}(x) = \exp\{-x^\gamma\}$, $x > 0$, $\gamma > 0$. $T = -\log \bar{F}(t) = t^\gamma$, $y = -\log \bar{F}(x) = x^\gamma$, $x = y^{1/\gamma}$, $dx = \gamma^{-1}y^{1/\gamma-1}dy$. For the first CTE, from (21),

$$
\mathbb{E}[X_1 | X_2 > t] = \gamma^{-1} \int_0^\infty e^{T} \hat{C}(e^{-y}, e^{-T})y^{1/\gamma-1}dy = \gamma^{-1}T^{1/\gamma} \int_0^\infty e^{T} \hat{C}(e^{-sT}, e^{-T})s^{1/\gamma-1}ds.
$$

Let $g(s; T) = 1 + T^{-1} \log \hat{C}(e^{-sT}, e^{-T})$. Then $g(\cdot; T)$ is decreasing in $s$, $g(0; T) = 0$ and $g(\infty; T) = -\infty$. For the second CTE, from (22),

$$
\mathbb{E}[X_1 | X_2 = t] = \gamma^{-1} \int_0^\infty \hat{C}_{1|2}(e^{-y}|e^{-T})y^{1/\gamma-1}dy = \gamma^{-1}T^{1/\gamma} \int_0^\infty \hat{C}_{1|2}(e^{-sT}|e^{-T})s^{1/\gamma-1}ds.
$$

Let $g(s; T) = T^{-1} \log \hat{C}_{1|2}(e^{-sT}|e^{-T})$. Then $g(\cdot; T)$ is decreasing in $s$, $g(0; T) = 0$ and $g(\infty; T) = -\infty$.

**Example 4 (Gumbel or extreme value copulas)** Approximations of (21) and (22) are simpler when $\hat{C}$ is the Gumbel copula and more generally a bivariate extreme value copula because $\hat{C}(v^*, v)$ has a closed form for $0 < v < 1$ and $s > 0$. From (22), we take $\hat{C}(u, v) = \exp\{-A(-\log u, -\log v)\}$, where $A(x_1, x_2)$ is homogeneous of order 1. In the proof of Proposition 2, $\hat{C}_{1|2}(u|v) = \hat{C}(u, v)A_2(-\log u, -\log v)v^{-1}$, where $A_2 = \frac{\partial A}{\partial x_2}$ is homogeneous of order 0.

Hence $\hat{C}(v^*, v) = v^A(s, 1)$ and $\hat{C}_{1|2}(v^*|v) = v^A(s, 1)-1A_2(s, 1)$. For the first CTE, log $\hat{C}(e^{-sT}, e^{-T}) = -TA(s, 1)$ and $g(0; T) = 1 - A(s, 1) - s$ plus the dominating part of $-\log f(F^{-1}(1 - e^{-sT}))$. For the second CTE, log $\hat{C}_{1|2}(e^{-sT}|e^{-T}) = -T[A(s, 1) - 1] + \log A_2(s, 1)$, and $g(s; T) = 1 - A(s, 1) - s$ plus the dominating part of $-\log f(F^{-1}(1 - e^{-sT}))$ and $h(s)$ includes $A_2(s, 1)$. For the Gumbel copula with parameter $\delta > 1$, $A(x_1, x_2) = (x_1^{\delta} + x_2^{\delta})^{1/\delta}$, $A(s, 1) = (s^{\delta} + 1)^{1/\delta}$, $A_2(x_1, x_2) = (x_1^{\delta} + x_2^{\delta})^{1/\delta-1}x_2^{\delta-1}$, $A_2(s, 1) = (s^{\delta} + 1)^{1/\delta-1}$.

Further concrete examples are given below with both $\bar{F}$ and $\hat{C}$ specified.

**Example 5 (Weibull margin, survival Gumbel copula)** For $\bar{F}$ being Weibull, and $\hat{C}$ being the Gumbel copula, we can illustrate the steps. The calculations make use of Examples 3 and 4. For Gumbel copula, with $\delta > 1$, then from Example 4, $\hat{C}(u^*, u) = u^{(1+s^{\delta})^{1/\delta}}$ and $g(s; T) = 1 - (1 + s^{\delta})^{1/\delta}$ does not depend on $T$. Let $z = (1 + s^{\delta})^{1/\delta} - 1$ and $s = m(z) = [(z + 1)^{\delta} - 1]^{1/\delta}$. $m'(z) = [(z + 1)^{\delta} - 1]^{1/\delta-1}(z + 1)^{1/\delta}$. Hence based on Example 3

$$
\mathbb{E}[X_1 | X_2 > t] = \gamma^{-1}T^{1/\gamma} \int_0^\infty e^{-Tz}[m(z)]^{\gamma-1}m'(z)dz = \gamma^{-1}T^{1/\gamma} \int_0^\infty e^{-Tz}[(z + 1)^{\delta} - 1]^{1/\delta-1}(z + 1)^{1/\delta}dz.
$$

Let $\Upsilon(z) = [(z + 1)^{\delta} - 1]^{1/\delta-1}(z + 1)^{1/\delta-1}$. This is bounded by an exponential function over all $z > 0$ and it is analytical for $0 < z < 1$, behaving like $(\delta z)^{1/\delta-1}$ for $z$ near 0. By a slightly modified version of Watson’s lemma (pp 20–21 of Wong (1989)), or Theorem 36 of Breitung (1994) the integral behaves like:

$$
\Gamma(\gamma^{-1} - 1)\gamma^{-1}T^{1/\gamma}T^{1/\gamma-1}T^{-1/\gamma}(z + 1)^{1/\delta-1} = O(t^{1-1/\delta}), \quad t \to \infty.
$$

(23)

For the Gumbel copula, with $\delta > 1$, from Example 4 $\hat{C}_{1|2}(u^*|u) = (1 + s^{\delta})^{1/\delta-1}u^{(1+s^{\delta})^{1/\delta}-1}$. Hence based
on Example 3

\[ \mathbb{E}[X_1|X_2 = t] = \gamma^{-1}T^{1/\gamma} \int_0^\infty e^{-T[(1+s^\delta)1/\delta-1](1+s^\delta)1/\delta-1}s^{1/\gamma-1}ds \]

\[ = \gamma^{-1}T^{1/\gamma} \int_0^\infty e^{-Tz[(z+1)^\delta-1]1/(\gamma\delta)-1}dz. \]

By Watson’s lemma, the integral behaves like (23).

As the upper order for a bivariate survival Gumbel copula is \( \kappa_U = 2^{1/\delta} \), a larger \( \delta \) implies a stronger degree of upper intermediate tail dependence. This observation is consistent to the pattern of \( O(t^{1-1/\delta}) \) for \( \mathbb{E}[X_1|X_2 > t] \) and \( \mathbb{E}[X_1|X_2 = t] \). As \( \delta \to 1 \) for the independence copula the rate is \( O(1) \), and as \( \delta \to \infty \) for comonotonicity the rate is \( O(t) \).

**Example 6 (Pareto margin, survival Gumbel copula)** Suppose \( X_1 \) and \( X_2 \) have a copula \( C \) such that \( \hat{C} \) is a Gumbel copula with parameter \( \delta > 1 \), so that the upper tail order of \( C \) is \( 1 < \kappa = 2^{1/\delta} < 2 \). From Examples 2 and 4, \( T = \alpha \log(1+t), e^{-sT}[f(F^{-1}(1-e^{-sT}))]^{-1} = \alpha^{-1}e^{sT/\alpha} \), and

\[ \mathbb{E}[X_1|X_2 > t] = \alpha^{-1}T \int_0^\infty e^{T\hat{C}(e^{-sT},e^{-T})}e^{sT/\alpha}ds \]

\[ = \alpha^{-1}T \int_0^\infty \exp\{T[1+s/\alpha - (s^\delta + 1)^{1/\delta}]\}ds \]

\[ =: \alpha^{-1}T \int_0^\infty e^{Ts\max\{g(s),z\}ds,} \]

where \( g(s) := 1 + s/\alpha - (s^\delta + 1)^{1/\delta} \). In order to apply the Laplace approximation as \( T \to \infty \), the function \( g \) has to satisfy some regularity conditions (see [Small, 2010]) as follows.

(1) Clearly, \( g(s) \) is differentiable on \((0, \infty)\), and the global maximum is attained when \( s = s_0 = [\alpha^{\delta/(\delta-1)} - 1]^{-1/\delta} \). As \( \alpha, \delta > 1 \), \( 0 < z_0 < \infty \). While \( g''(s) = (1-\delta)\alpha s^{\delta-2}(1+s^{\delta})^{1/\delta-2} < 0 \) for \( s \in (0, \infty) \) when \( \alpha, \delta > 1 \).

(2) Since \( g(0) = 0 \), \( g(\infty) = -\infty \), \( g(s) \) is strictly increasing for \( s \in (0, z_0) \) and \( g(s) \) is strictly decreasing for \( s \in [s_0, \infty) \), we may choose a \( 0 < \xi < s_0 \) and \( \epsilon = g(s_0) - \max\{g(s_0-\xi), g(s_0+\xi)\} > 0 \) such that \( g(s) < g(s_0) - \epsilon \) for all \( s \in (0, \infty) \cap \{ s : |s - s_0| \geq \xi \} \).

(3) Also, \( \int_0^\infty \exp\{1 + \alpha^{-1}s - (s^\delta + 1)^{1/\delta}\}ds < e \int_0^\infty e^{s(1/\alpha-1)}ds < \infty \).

Therefore, as \( t \to \infty \),

\[ \mathbb{E}[X_1|X_2 > t] \sim \alpha^{-1}T e^{Ts\max\{g(s_0),z\} \sqrt{\frac{2\pi}{-Tg''(s_0)}}} = (1+t)^{\alpha\max\{g(s_0),z\}} \sqrt{\frac{2\pi\alpha \log(1+t)}{-g''(s_0)}}, \]

and \( \alpha g(s_0) = \alpha - \alpha^{\delta/(\delta-1)} - (\delta-1)/\delta \) > 0 affects the behavior of \( \mathbb{E}[X_1|X_2 > t] \).

Let \( \xi(\alpha, \delta) := \alpha g(s_0) \), then \( \partial \xi(\alpha, \delta)/\partial \alpha = 1 - (1 - \alpha^{\delta/(\delta-1)})^{-1/\delta} < 0 \) and thus \( \xi(\alpha, \delta) \) is decreasing in \( \alpha \). So increasing \( \alpha \), the parameter for the Pareto margins, will decrease the speed of \( \mathbb{E}[X_1|X_2 > t] \) as \( t \to \infty \). Similarly, it can be easily verified that, for \( \alpha, \delta > 1 \), \( \partial \xi(\alpha, \delta)/\partial \delta > 0 \). Therefore, increasing \( \delta \) will increase the speed of \( \mathbb{E}[X_1|X_2 > t] \) as \( t \to \infty \). Those are consistent to the plots in Figure 4. Note that, the speed of \( \mathbb{E}[X_1|X_2 > t] \) can be roughly compared by comparing the relative positions of the CTE lines to 45\(^{\circ}\) degree.
line. Next, we approximate (22) and use Example 4:

$$
E[X_1|X_2 = t] = \alpha^{-1}T \int_0^\infty \exp \left\{ T \left[ 1 + s/\alpha - (s^\delta + 1)^{1/\delta} \right] \right\} \times (1 + s^\delta)^{1/\delta - 1} ds,
$$

where $g(s) := 1 + s/\alpha - (s^\delta + 1)^{1/\delta}$ (same as above) and $h(s) := (1 + s^\delta)^{1/\delta - 1}$. We may also use a Laplace approximation. The function $h(s)$ is continuous on $(0, \infty)$, and $h(s_0) > 0$. Therefore (see Small [2010]), as $t \to \infty$,

$$
E[X_1|X_2 = t] \sim \alpha^{-1}Te^{Tg(s_0)}h(s_0)\sqrt{\frac{2\pi}{-Tg''(s_0)}} = (1 + t)^{\alpha g(s_0)}h(s_0)\sqrt{\frac{2\pi \log(1 + t)}{-g''(s_0)}}.
$$

## 4 Discussion and future research

Although we assume that the univariate margins are identical to simplify the notation, the conditions such as finite first moment and the margins belonging to either MDA(Fréchet) or MDA(Gumbel) are only for $X_1$. One does not need to specify conditions on the conditioning random variable $X_2$ as long as it is supported on $[x, \infty)$ with a real number $x$ and satisfies the regularity condition $\limsup_{t \to \infty} \int_t^\infty F_1(x) dx / |tF_2(t)| < \infty$ in the sense of Proposition 8. Moreover, $X_2$ can even follow a discrete distribution. For example, $X_2$ can be supported on the non-negative integers, such as negative binomial with exponentially decreasing tail or discretized Pareto with regularly varying tail; in this case, the limiting behavior as in Section 3 is along integers $t \to \infty$. The results in Section 3 on CTE extend to both tails for distributions with support on all of the real line; the derivations involve splitting $X_1$ into two parts $X_1 = X_1^+ - X_1^- = \max\{X, 0\} - \max\{-X, 0\}$, and then the tail behavior of $F(x)$ as $x \to -\infty$ needs also be considered.

For Markov time series based on copulas, arbitrary univariate margins can be used, and then the results distinguish the conditional tail expectations for different copula models. For these models, we also want to know the asymptotic form of the conditional tail variance. For the middle of the range of support, the conditional tail expectations or the conditional tail variances are similar for different bivariate copulas, but the tails can differ a lot. The choice of the copula can has a big influence for predictions conditioned on extreme values.

For time series, Biller and Nelson (2005) has models based on Gaussian autoregressive processes transformed to other margins, that is $F^{-1}(\Phi(Y_t))$ where \{Y_t\} is a Gaussian autoregressive time series, scaled to standard normal margins with cdf $\Phi$. So this is a special case of Markov time series based on a copula model for consecutive observations. The advantage of this time series modeling approach over those based on thinning operators Joe 1996 Weiβ 2008 is that extensions to accommodate covariates is easier and also there is more flexibility in the margins. For the bivariate normal or Gaussian copula, we know that the conditional expectation is linear and the conditional variance is constant when the univariate margin is normal. For other margins, the procedures in Section 3.4 can be used to numerically get the asymptotic rate of increase (for positive correlation) by approximating the function $g(s_0(T), T)$, which appears after (21) and (22), for large $T$. Analytical derivation of this rate seems intractable, but numerical plots suggest that the asymptotic rate.
is sublinear and sometimes nearly linear.

The procedures introduced in Section 3.4 work for checking the tail behavior of the CTEs for specified margins and copulas. More general theory regarding the role of the tail order function [Hua and Joe (2011)] in determining the tail pattern of the form $O(t^\gamma)$ with $0 < \gamma < 1$ for those CTEs is also interesting, since the theory may help us further develop semi-parametric approaches for multivariate tail assessment.

For applications to coherent risk measures with multiple risks, we would like to extend to multivariate copulas so that we can get CTE for one risk given that others are large. This will require multivariate extensions of the results on boundary conditional cdfs.

We have shown in Figure 1 some CTE plots for different bivariate copulas. With a bivariate data, one could transform each variable to a suitable univariate margin and then get empirical versions of $\mathbb{E}[X_1 | X_2 > t]$ and $\mathbb{E}[X_1 | X_2 = t]$ as functions of $t$: the former is usually easier to obtain. Our interest is in whether one of these empirical CTE plots can be used to diagnose the strength of dependence in the tails and hence help in the choice of bivariate copulas, say in a vine construction; the idea here is in parallel to using the mean excess function (e.g., Embrechts et al. 1997) to detect the tail pattern of univariate margins.

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